α MATHEMATICS

June 2024 **Time: 3 hours**

Grade 12 **Total: 200 marks**

INSTRUCTIONS AND INFORMATION

Please read the following instructions carefully before answering any questions:

1. This question paper consists of six pages, a formula sheet of three pages and an answer sheet of four pages.
2. Answer all 11 questions.
3. Non-programmable calculators may be used, unless otherwise indicated.
4. Unless indicated otherwise, all answers, where applicable, must be given correct to **two decimal places**.
5. All necessary calculations must be shown. The correct answer on its own will not necessarily lead to full marks.
6. All angles are given in radians. Answers must be given in radians where applicable.
7. The diagrams in the question paper are not necessarily drawn to scale.
8. Write neatly and legibly.

Question 1 [30 marks]

- The following questions have only ONE correct answer.
- Only mark the correct letter, A, B, C or D with a X (cross) on the answer sheet provided.
- Each question counts 2 marks.

1.1 If $z^2 = 9\text{cis}\left(\frac{\pi}{6}\right)$, what will the value(s) of z be?

(A) $z = 3\text{cis}\left(\frac{\pi}{12}\right)$ **(B)** $z = 3\text{cis}\left(\frac{\pi}{3}\right)$

(C) $z = 3\text{cis}\left(\frac{\pi}{12}\right)$ or $z = 3\text{cis}\left(\frac{13\pi}{12}\right)$ **(D)** $z = 3\text{cis}\left(\frac{\pi}{3}\right)$ or $z = 3\text{cis}\left(\frac{13\pi}{3}\right)$

1.2 Solve for$x$ if $-\frac{|2x+1|}{3} < 1$

(A) $x < -2$ or $x > 1$ **(B)** $x \in \mathbb{R}$

(C) $-2 < x < 1$ **(D)** No solution

1.3 Which of the following equations will be the asymptote of $f(x) = e^{3x-1} - 2$?

(A $x = \frac{1}{3}$ **(B)** $y = 2$

(C) $x = -\frac{1}{3}$ **(D)** $y = -2$

1.4 The quotient of $2\text{cis}\left(\frac{\pi}{3}\right)$ and $6\text{cis}\left(\frac{-\pi}{6}\right)$ is

(A) $\frac{1}{3}\text{cis}\left(\frac{\pi}{2}\right)$ **(B)** $\frac{1}{3}\text{cis}\left(\frac{\pi}{6}\right)$

(C) $3\text{cis}\left(\frac{\pi}{2}\right)$ **(D)** $3\text{cis}\left(\frac{\pi}{6}\right)$

1.5 In the expansion of $\frac{1}{1-x}$, what will the fourth term be?

(A) $\frac{x^3}{6}$ **(B)** x^3

(C) $\frac{-x^3}{6}$ **(D)** $-x^3$

1.6 Two vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ will be perpendicular if

(A) $\boldsymbol{p} \bullet \boldsymbol{q} = 0$ **(B)** $\boldsymbol{p}$ x $\boldsymbol{q} = 0$

(C) $\boldsymbol{p} \bullet \boldsymbol{q} \neq 0$ **(D)** $\boldsymbol{p}$ x $\boldsymbol{q} \neq 0$

1.7 The x-coordinate(s) of the stationary point of $f(x) = x + \frac{1}{x}$ is

(A) $x = 0$ **(B)** $x = -1$

(C) $x = 1$ **(D)** $x = \pm 1$

1.8 If M is a 2x3-matrix and N is a 4x2-matrix, which of the following operations is possible?

(A) MN **(B)** $M + N$

(C) NM **(D)** $M - N$

1.9 If a function has a local maximum at $x = 2$, which of the following is true?

(A) $f(2) > 0$ **(B)** $f''(2) > 0$

(C) $f(2) < 0$ **(D)** $f''(2) < 0$

1.10 For which values of x will the expansion of $\sqrt{3 + x}$ converge?

(A) $|x| < \frac{1}{3}$ **(B)** $|x| > \frac{1}{3}$

(C) $|x| < 3$ **(D)** $|x| > 3$

1.11 Determine the oblique asymptote of $y = \frac{x^2+x-3}{x}$

(A) $y = x + 1$ **(B)** $y = x$

(C) $y = x^2 + x - 3$ **(D)** Not one of the above

1.12 Write as one logarithm: $y = \ln x - 5\ln y - \ln z$

(A) $y = \ln\left(\frac{xz}{y^5}\right)$ **(B)** $y = \ln\left(\frac{x}{y^5 z}\right)$

(C) $y = \ln\left(\frac{x}{5yz}\right)$ **(D)** $y = \ln\left(\frac{xz}{5y}\right)$

1.13 With which formula can the area under the curve $y = \sec^2\left(\frac{x}{4}\right)$ be determined?

(A) $\frac{\tan\left(\frac{x}{4}\right)}{4} + c$ **(B)** $\frac{-\text{ta}\ \left(\frac{x}{4}\right)}{4} + c$

(C) $4\tan\left(\frac{x}{4}\right) + c$ **(D)** $-4\tan\left(\frac{x}{4}\right) + c$

1.14 The function f has a vertical asymptote at $x = 1$. Which of the following is true?

(A) $\lim_{x\to 1} f = 0$ **(B)** $\lim_{x\to 1} f = \infty$

(C) $\lim_{x\to \infty} f = 0$ **(D)** $\lim_{x\to \infty} f = 1$

1.15 If $f(x) = \ln x$ and $g(x) = e^{3x}$, determine the value of $(f \circ g)(2)$.

(A) 8 **(B)** 3

(C) 2 **(D)** 6

Question 2 [20 marks]

2.1 Given: $p = -1 + \sqrt{3}i$ and $k = 2e^{-\frac{\pi i}{4}}$

a) Convert p and k into polar form. (4)

b) Determine $p.k$ (2)

c) Use De Moivre's theorem and determine p^3. (2)

d) Determine a and b if $p^3 = a + bi$. (3)

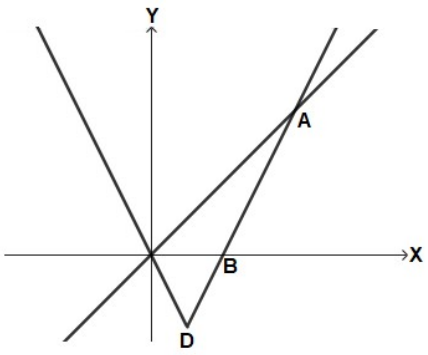
2.2 a) Show that $x = i$ is a zero of $m(x) = x^4 + x^3 - 5x^2 + x - 6$ (3)

b) If $x^2 + 1$ is a factor of m, factorise m fully in $\mathbb{C}[x]$. (6)

Question 3 [17 marks]

3.1 Given: $ax - 2y = 5$ and $-x + ay = -4$
Use Cramer's rule and determine the value of a if $x = 1$. (7)

3.2 The graphs of $f(x) = |2x - 1| - 1$ and $g(x) = x$ is given.



a) Determine the x-coordinate of B, one of the x-intercepts of f. (2)

b) The graphs intersect at $(0; 0)$ and A. Determine the coordinates of A. (3)

c) Determine the coordinate of D, the salient point of f. (2)

d) Solve for x if $f(x) < 4$ (3)

Question 4 [15 marks]

4.1 Determine $f(-3)$ if $f(x) = \arccos(e^{2x+5})$ (2)

4.2 Solve for x: $\ln(e^{2x} - 12) = x$ (6)

4.3 Determine the inverse of $y = \frac{e^{3x}}{2} + 1$ (4)

4.4 Sketch the graph of $y = 2\ln(x-1)$ on the given answer sheet.
Show all intercepts with the axis and the asymptotes (if applicable) clearly. (3)

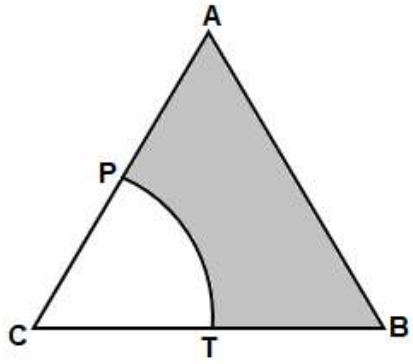
Question 5 [15 marks]

5.1 Determine the coefficient of x^{-2} in the binomial expansion of $\left(x + \frac{2}{x^2}\right)^{10}$. (6)

5.2 Use mathematical induction and prove that $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$
for all $n \in \mathbb{N}$: (9)

Question 6 [17 marks]

6.1 The diagram shows an equilateral triangle ABC with side lengths of 4cm. P and T are the midpoints of AC and CB. CPT is a sector of a circle with centre C.



a) Determine the circumference of CPT. Give your answer in terms of π. (2)

b) Determine the area of sector CPT. Give your answer in terms of π. (2)

c) Hence, determine the area of the shaded area PABT.
Give your answer as a decimal number. (4)

6.2 Given vectors $\boldsymbol{a} = 2i - j - 2k$ and $\boldsymbol{b} = i + j$

a) Determine the unit vector of $\boldsymbol{a}$. (3)

b) Determine the angle of vector $\boldsymbol{a}$ with the z-axis. (2)

c) Determine two vectors perpendicular to the plane through $\boldsymbol{a}$ and $\boldsymbol{b}$. (4)

Question 7 [14 marks]

7.1 Given: $f(x) = \arctan(x+1) + \frac{\pi}{2}$

Use the answer sheet and draw the graph of f.
Show all intercepts with the axis and the asymptotes (if applicable) clearly. (4)

7.2 Given: $f(x) = \sin\left(x - \frac{\pi}{6}\right)$, $\frac{-\pi}{2} < x < \frac{\pi}{2}$

a) Determine the coordinates of the inflection point of f. (5)

b) Fully motivate why your answer in (a) is an inflection point. (5)

Question 8 [17 marks]

8.1 Use the answer sheet and draw examples of the following: (8)

a) A jump discontinuity

b) Two different examples of a removable discontinuity

c) A function which is continuous, but not differentiable

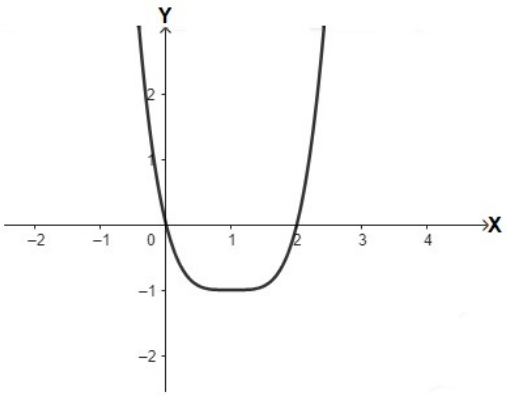
8.2 Differentiate the following:

a) $f(x) = \tan^3(x^2) - \frac{1}{7x}$ (4)

b) $y = \arctan(2x) + \log(\sin x)$ (5)

Question 9 [18 marks]

9.1 The function f'' is given. The function has x-intercepts at $x = 0$ and $x = 2$ and a stationary point at $x = 1$.



Give all the x-values where the following is true:

a) The function f bends concave up.

b) The function f' decreases.

c) The function f' has a stationary point. (6)

9.2 a) Use implicit differentiation and determine $\frac{dy}{dx}$ if $e^y\sqrt{x} = ey^2$ (8)

b) Hence, determine the equation of the tangent at $(1; 1)$. (4)

Question 10 [19 marks]

10.1 Given: $f(x) = \frac{x^2+x+1}{x^2}$

a) Determine the asymptotes of the function and the nature thereof. (4)

b) The function intersects one of the asymptotes. Determine the x-coordinate of this intercept. (3)

c) Determine the x-value(s) where f increases. (4)

10.2 a) Differentiate $y = 2^{3x-1} + \ln(x+1)$ (4)

b) Hence, use **Newton's method** and determine the x-intercept, correctly to five decimal places, of $f(x) = 2^{3x-1} + \ln(x+1)$ and $g(x) = \ln 2$.
Use $x = 0{,}2$ as a first approximation. Clearly show how you use Newton's method. (4)

Question 11 [18 marks]

11.1 Simplify:

a) $\int (5^{7x} - e^{2x})dx$ (5)

b) $\int \left(\frac{3}{(4x-1)^2} + \frac{1}{(x+1)}\right) dx$ (4)

c) $\int \left(\frac{e}{1+x^2} - \frac{1}{\sqrt{x}} + \ln 5\right) dx$ (3)

11.2 Find the volume of the rotating body formed when the curve $y = \sqrt{x+\pi}$ is rotated about the x-axis, between $x = p$ and $x = 0$. $(p < 0)$.
Give your answer in terms of p. (6)

Total: 200