α-WISKUNDE

Alpha Mathematics
FINAL EXAM PAPER

23 October 2023 **Time: 3 hours**

Grade 12 **Total: 200 marks**

INSTRUCTIONS AND INFORMATION

Carefully read through the following instructions before answering the examination paper:

1. Answer all 10 questions on this examination paper.
2. Write your name and ID number on the front page of the examination paper.
3. Non-programmable calculators may be used, unless otherwise indicated at a specific question.
4. Unless indicated otherwise, all answers, where applicable, must be given correct to two decimal places.
5. The diagrams in the examination paper are not necessarily drawn to scale.
6. All angles are given in radians. Answers must be given in radians where applicable.
7. This examination paper consists of a front page, 21 pages and a formula sheet of 3 pages.
8. Question 1 consists of 10 multiple choice questions. Answer it on the answer sheet. This answer sheet is at the front of the paper.
   Do not remove the answer sheet from the examination paper.
9. For all other questions, all necessary calculations must be shown clearly. The correct answer on its own will not necessarily lead to full marks.
10. Additional writing space is provided at the end of this examination paper. Clearly indicate if you made use of this to complete a question.
11. Write neatly and legibly.

Question 1 [30 Marks]

- Answer this question **on the answer sheet,** which is attached to the front, by making an X (cross) on A, B, C or D. These questions count 2 marks each.
- Please **DO NOT** detach this page from the paper.

1.1 Twee vectors are parallel to each other if

(A) the point product between them is equal to zero.

(B) the cross product between them is equal to zero.

(C) the point product between them is equal to one.

(D) the cross product between them is equal to the unit vector.

1.2 What is usually calculated with the Newton-Raphson method?

(A) x- intercepts of the tangents. **(B)** y- intercepts of the tangents.

(C) The gradient of the tangents. **(D)** The determinant of the matrix.

1.3 If $2x + 3$ is a zero of the polynomial $P(x)$, according to the factor theorem:

(A) $P\left(\frac{3}{2}\right) = 0$ **(B)** $P\left(-\frac{3}{2}\right) = 0$

(C) $P\left(\frac{2}{3}\right) = 0$ **(D)** $P\left(-\frac{2}{3}\right) = 0$

1.4 Vectors u, v and w are given with $u \times v = w$. If $u = (2; 3; -1)$, $v = (0; a; b)$ and $w = (17; -10; -4)$, determine the value of b.

(A) 5 **(B)** -2

(C) -4 **(D)** -5

1.5 Which one of the following sets of equations can be solved by using Cramer's rule?

(A) $2x - 5y + 4z = 5$
$x + 6y - 2z = -7$

(B) $5x + 2y = -2$
$-2x + y = -9$
$x - 3y = 7$

(C) $\frac{2}{3}x - 5y = 10$
$2x - 15y = 2$

(D) $x + y - z = 6$
$3x - 2y + z = -5$
$x + 3y - 2z = 14$

1.6 Given the polynomial equation $P(x) = x^3 + 6x^2 - 5x + 1$. At which one of the following x-values will the polynomial be concave down?

(A) 1 **(B)** -3

(C) -2 **(D)** not one of these

17 The coefficient of the variable of the 4^{th} term in the power series of $\frac{1}{1-2x}$ is equal to:

(A) $-\frac{8}{3}$ **(B)** 16

(C) -8 **(D)** 8

1.8 The solution of an absolute value inequality is equal to $-2 < x < 6$. Which of the following inequalities will give this solution?

(A) $|2 - x| < 4$ **(B)** $|2 - x| > 4$

(C) $|4 - x| < 2$ **(D)** $|4 - x| > 2$

1.9 The function $f(x) = \arcsin(2x - 1)$ has a domain of:

(A) $0 \leq x \leq \frac{\pi}{2}$ **(B)** $-\frac{1}{2} \leq x \leq \frac{1}{2}$

(C) $0 \leq x \leq 1$ **(D)** $0 \leq x \leq \pi$

1.10 If the radius of a circle equals 5 cm, which one of the following is NOT a possible value of the length of the arc?

(A) 2π cm **(B)** 30 cm

(C) 25 cm **(D)** 32 cm

1.11 If P is a polynomial of degree n with $n > 0$, what will the degree of $\int_0^x P(t)\,dt$ be?

(A) $n - 1$ **(B)** n

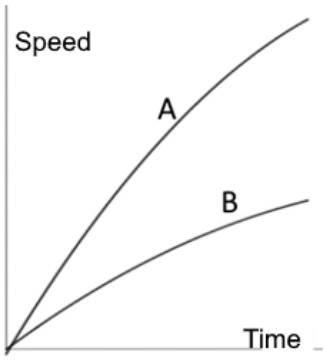
(C) $n + 1$ **(D)** 1

1.12 Simplify: $\frac{8\text{cis}(\frac{3\pi}{4})}{2\text{cis}(-\frac{\pi}{4})}$

(A) $-4i$ **(B)** 4

(C) -4 **(D)** $4i$

1.13 If $f(t)$ is a function that represents the distance of an object from a point, then the derivative of the function, $f'(t)$, will give the speed of the object.



This sketch represents the speed in km/h of two vehicles A and B. The vehicles depart simultaneously from the same point on a straight road. What will the area between the two graphs represent at t = 2 hours?

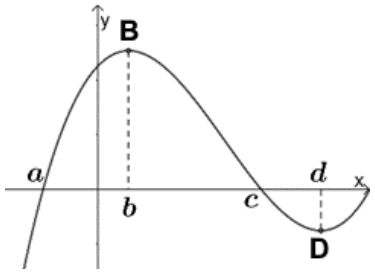
(A) The distance between the two vehicles.

(B) The difference in the speed of the two vehicles.

(C) The sum of the speed of the two vehicles.

(D) The combined distances covered by the two vehicles.

1.14 The sketch shows the graph of a polynomial function. B is the local maximum turning point and D the local minimum turning point. For which value of x is it true that $f''(x) < f'(x) < f(x)$?



(A) a **(B)** b **(C)** c **(D)** d

1.15 If $\frac{dy}{dx} = y(1 + \ln x)$ then $y = \ldots$

(A) $e^{(\frac{x^2}{2}+\ln x)}$ **(B)** $e^{1+\ln x}$

(C) $(1 + \ln x)^2$ **(D)** $e^{x\ln x}$

Answer the following questions **on the exam paper** on the lines provided after each question. Clearly indicate if you use the additional writing space at the end of the paper to complete a question.

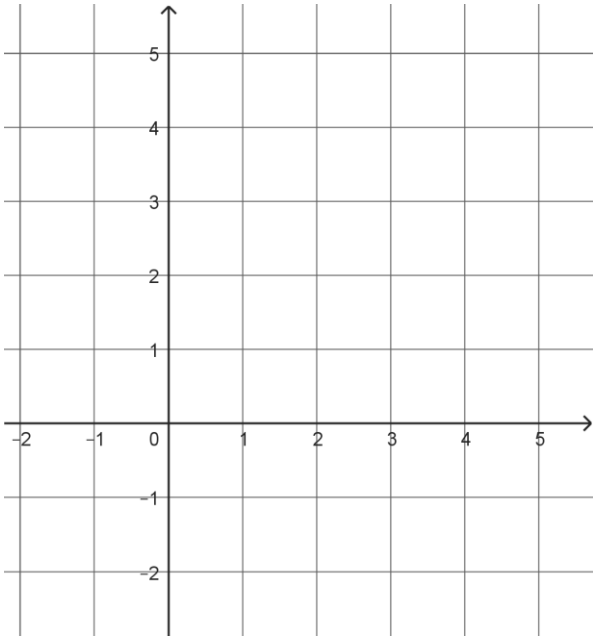
QUESTION 2 **[20 MARKS]**

2.1 Given the function: $f(x) = e^{x-2} + 2$.

(a) Determine the value of $f(2)$. (1)

(b) Determine the inverse function and write it as $f^{-1}(x) = \cdots$ (2)

(c) Draw, on the same set of axes, sketch graphs of f and f^{-1}. Show the value calculated in 2.1 (a) on both graphs. (6)



(d) Describe the transformations that will form $h(x) = -e^{x-2} + 4$ from f. (2)

(e) Use Newton's method and determine the x-value of the point where h and $f^{-1}(x) = \ln(x-2)+2$ intersect. Use $x = 3$ as first approximation and give the answer correct to 4 decimal numbers. (5)

2.2 Given $f(x) = e^{2x} - 16$ and $g(x) = \ln(x-2)$. Solve for x if $f(g(x)) = 0$. (4)

QUESTION 3 **[16 MARKS]**

3.1 (a) Write $(-1+\sqrt{3}i)$ in polar form. Use π and root form where possible. (3)

(b) Calculate the value of $\left(-1+\sqrt{3}i\right)^7$. Do the calculation in polar form and give the answer in rectangular form. (3)

(c) Determine all the values or z if $z^2 = 8(-\sqrt{3}-i)$. Leave the answer(s) in polar form. (5)

3.2 The equation $2x^3+7x^2+22x-13=0$ has a root $x=-2+3i$. Determine the real root of the equation. (5)

QUESTION 4 **[20 MARKS]**

4.1 Solve for x if $x(|x| - 3) = -2$ (7)

4.2 Use mathematical induction and prove that: $4 + 9 + \cdots + (5n - 1) = \frac{5n^2+3n}{2}$ (8)

4.3 Use the binomial theorem and calculate the sixth term in the expansion of:

$$\left(2x - \frac{1}{3x}\right)^{12}$$ (5)

QUESTION 5 **[18 MARKS]**

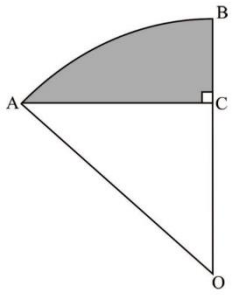
5.1 The vectors $i + aj + 6k$ and $4i + 2j - 3k$ are perpendicular to each other. Determine the value of a. (2)

5.2 Determine the angle between the vector $4i + 2j - 3k$ and the y-axis. (2)

5.3 a) Determine a vector that will be perpendicular to the vectors $\boldsymbol{a} = (1; 2; -4)$ and $\boldsymbol{b} = (2; -1; -3)$ (5)

b) Hence calculate the area of the parallelogram formed by $\boldsymbol{a}$ and $\boldsymbol{b}$. (2)

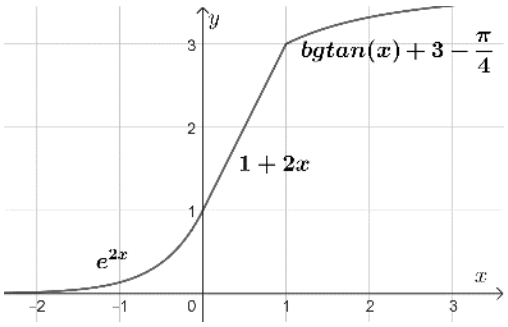
5.4 The sketch beneath shows a sector shaped courtyard OAB with midpoint O and radius r. AC is perpendicular to OB and $A\hat{O}B = \frac{\pi}{6}$ radians. The shaded region must be paved. The area of this region is $\frac{2\pi - 3\sqrt{3}}{6}\ m^2$. Calculate the value of r. (7)



QUESTION 6 **[19 MARKS]**

6.1 The following is the equation of the adjacent sketch:

$$f(x) = \begin{cases} e^{2x} & \text{if} \quad x \leq 0 \\ 1 + 2x & \text{if} \quad 0 < x < 1 \\ \arctan(x) + 3 - \frac{\pi}{4} & \text{if} \quad x \geq 1 \end{cases}$$



(a) Give the equation of the derivative f', as a piece-wise function. (3)

(b) The function f is continuous at $x = 0$. Determine algebraically if f is differentiable at $x = 0$. (3)

(c) Show algebraically that f is continuous at $x = 1$. (3)

(d) Determine algebraically if f is differentiable at $x = 1$. (3)

6.2 Differentiate the following functions:

(a) $f(x) = \frac{5^{\pi x}}{\ln(5x)+\pi}$ (4)

(b) $g(x) = \text{cosec}\sqrt{5x}$ (3)

QUESTION 7 **[21 MARKS]**

The graph of the function $f(x) = \frac{x^3-4x^2}{x^2-1}$ has only two stationary points, of which one is a local maximum turning point at $(-2{,}5;\ -7{,}7)$.

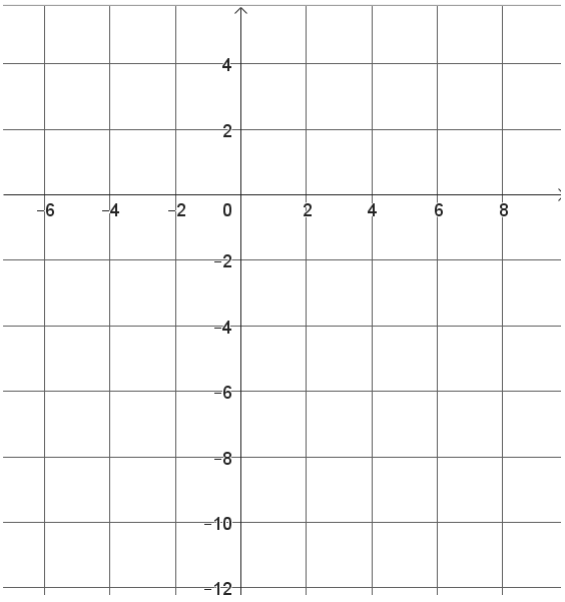
7.1 Calculate all intercepts of f with the axes. (3)

7.2 Determine the equations of all the asymptotes of f. (4)

7.3 Determine the point where f will intercept one of the asymptotes. (3)

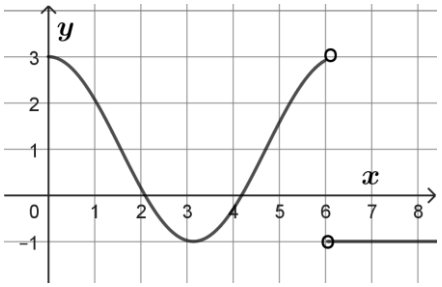
7.4 Calculate the coordinates of the local minimum turning point of f. (5)

7.5 Draw a neat sketch graph of $y = f(x)$. Clearly show the intercepts with the axes, the asymptotes and the turning points on your sketch. (6)



QUESTION 8 **[18 MARKS]**

8.1 The sketch shows the graph of the **derivative f'** of a **continuous** function f.

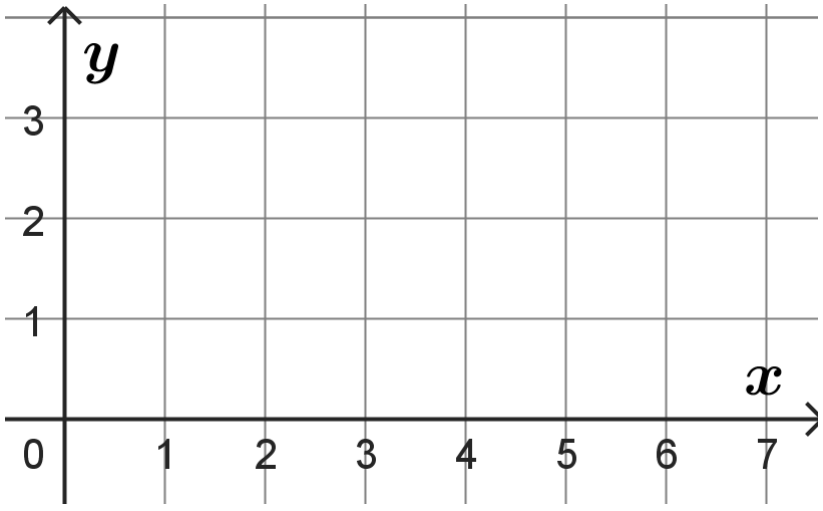


(a) Give the x-values of the stationary points of f. (2)

(b) Give, with motivation, the nature of these stationary points. (4)

(c) Give the x-value of the point of inflection of f if $x < 6$. (1)

(d) It is given that the graph of f starts at the point (0; 0). Draw a possible sketch graph of $y = f(x)$ for $x \in [0; 7]$. Accept that the graph does not intersect the x-axis again. *Do not calculate f.* (5)



8.2 Use implicit differentiation and determine the equation of the tangent to the function $x^3 + y^3 = 6xy$ at the point (3; 3). (6)

QUESTION 9 **[24 MARKS]**

9.1 Use a Riemann sum and determine the value of $\int_0^2 (3x^2 + x)\,dx$ (9)

9.2 Determine the following integrals:

(a) $\int \frac{2x-3}{(x-2)^2}\,dx$, by using partial fractions. (5)

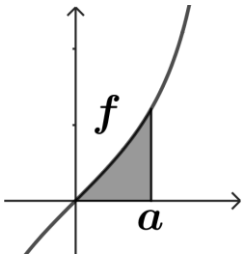
(b) $\int (2\sin^2 4x + 3^{\pi x})\,dx$ (5)

(c) $\int \frac{(\log x)}{x^2}\,dx$, use factor (piece-wise) integration. (5)

QUESTION 10 **[14 MARKS]**

10.1 The graphs of $y = 4 - x$ and $y = \sqrt{4 - x}$ intersect at $x = 3$ and $x = 4$. The area between the graphs rotate around the x-axis. Calculate the volume of the solid of revolution formed. Leave the answer in terms or π. Show all calculations. (7)

10.2 The sketch shows the graph of $f(x) = \frac{2\arcsin(x)}{\sqrt{1-x^2}}$



The shaded area is the region lies between the graph, the x-axis and the line $x = a$. The area of this region is equal to $\frac{\pi^2}{36}$. Calculate the value of a. (7)

TOTAL: 200 MARKS

EXTRA SPACE