

α MATHEMATICS

PRELIMINARY EXAM

September 2023

Grade 12

Time: 3 hours

Total: 200 marks

INSTRUCTIONS AND INFORMATION

Please read the following instructions carefully before answering any questions:

1. Answer all nine questions of this paper.
2. Write your name on the answer sheet.
3. Non-programmable calculators may be used unless otherwise indicated.
4. Unless indicated otherwise, all answers, where applicable, must be given correctly to **two decimal places**.
5. The diagrams in the question paper are not necessarily drawn to scale.
6. **All angles are given in radians.** Answers must be given in radians where applicable.
7. This question paper consists of six pages, a formula sheet of three pages and an answer sheet of three pages.
8. Question 1 consists of 15 multiple choice questions. Answer this on the answer sheet.
9. All necessary calculations must be shown. The correct answer on its own will not necessarily lead to full marks.
10. Write neatly and legibly.

QUESTION 1 [30 MARKS]

- The following questions have only ONE correct answer.
- Only mark the correct letter, A, B, C or D, with a X (cross) on the answer sheet provided.
- Each question counts 2 marks.

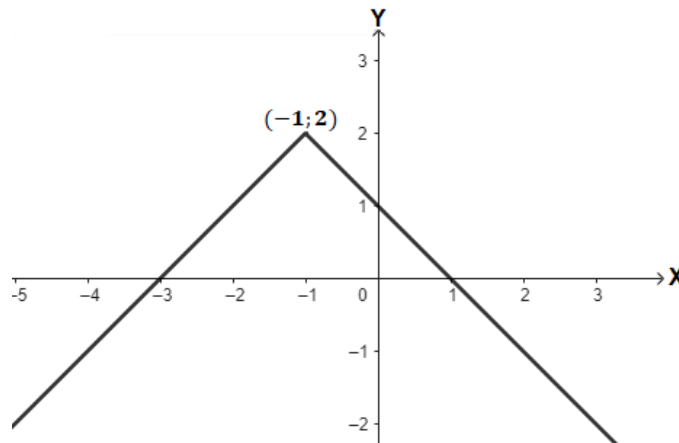
1.1 Given: $f(x) = e^{x-1} - 1$ Determine the asymptote and x -intercept of the function:

- (A) Asymptote: $y = -1$; x -intercept: $x = -1$
(B) Asymptote: $y = -1$; x -intercept: $x = 1$
(C) Asymptote: $y = 1$; x -intercept: $x = -1$
(D) Asymptote: $y = 1$; x -intercept: $x = 1$

1.2 Write $2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$ in rectangular form:

- (A) $-1 + \sqrt{3}i$ (B) $1 - \sqrt{3}i$
(C) $-\sqrt{3} + i$ (D) $\sqrt{3} - i$

1.3 The sketch of $y = f(x)$ is given below. The function has a salient point at $(-1; 2)$. The x -intercepts are at $x = -3$ en $x = 1$ and the y -intercept is at $y = 1$. Determine the equation of f .



- (A) $f(x) = |x + 1| + 2$ (B) $f(x) = -|x + 3| - 1$
(C) $f(x) = -|x + 1| + 2$ (D) $f(x) = -|x + 1| - 3$

1.4 The third term in the binomial expansion of $\sqrt[4]{1-2x}$ is

- (A) $-\frac{3x^2}{32}$ (B) $-\frac{1}{16}$
 (C) $-\frac{3x^2}{8}$ (D) $\frac{7x^3}{16}$

1.5 Determine the direction of the vector (2; 1) relative to the x -axis:

- (A) 0,46 rad (B) 0,52 rad
 (C) 1,05 rad (D) 1,11 rad

1.6 The asymptotes of $y = \arctan(2x - 1) + \frac{\pi}{4}$ are

- (A) $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ (B) $y = \frac{3\pi}{4}$ and $y = -\frac{\pi}{4}$
 (C) $y = \frac{\pi}{4}$ and $y = -\frac{3\pi}{4}$ (D) $y = \pi$ and $y = -\pi$

1.7 Given: $-2ax + y = -7$
 $5x - y = 8$

For which value of a will the system of equations have no solution?

- (A) $a = \frac{5}{2}$ (B) $a = \frac{2}{5}$
 (C) $a = \frac{-5}{2}$ (D) $a = \frac{-2}{5}$

1.8 Simplify: $\frac{2\left(\cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right)\right)}{6\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)}$

- (A) $3\text{cis}\left(\frac{15\pi}{4}\right)$ (B) $\frac{1}{3}\text{cis}\left(\frac{4\pi}{15}\right)$
 (C) $\frac{1}{3}\text{cis}\left(\frac{15\pi}{4}\right)$ (D) $3\text{cis}\left(\frac{4\pi}{15}\right)$

1.9 Given: $f(x) = \sqrt{x^2 - 1}$ and $g(x) = x + 1$. What is the value of $(f \circ g)(-1)$?

- (A) 1 (B) 0
 (C) -1 (D) i

1.10 Which one of the following formulas can be used to calculate the volume if $y = \cos(3x)$ rotates around the x -axis, between $x = a$ and $x = b$?

- (A) $\frac{\pi}{2} \int_a^b (1 + \cos(2x)) dx$ (B) $\pi \int_a^b (1 + \cos(2x)) dx$
 (C) $\frac{\pi}{2} \int_a^b (1 + \cos(6x)) dx$ (D) $\pi \int_a^b (1 + \cos(6x)) dx$

1.11 If $p(x) = x^4 - 2x^3 + 5x^2 - 6x + 6$, which of the following will be a factor of p , if $x = 1 - i$ is a zero of p ?

(A) $x^2 - 2x + 2$

(B) $x^2 + 2x + 2$

(C) $x^2 + 2x - 2$

(D) $x^2 - 2x - 2$

1.12 If $y = f(x)$ is differentiable at $x = 1$, which of the following options is true?

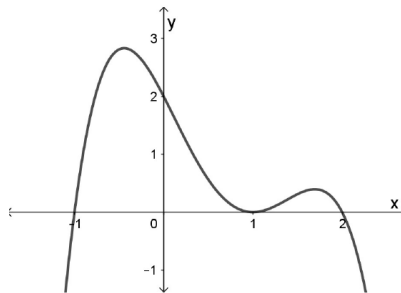
(A) $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x)$

(B) $\lim_{x \rightarrow 1} f(x) = f(1)$

(C) $\lim_{x \rightarrow 1^-} f'(x) = \lim_{x \rightarrow 1^+} f'(x)$

(D) All of the above options.

1.13 Given the following graphical representation of f . The function has turning points at $x = -\frac{1}{2}$, $x = 1$ and $x = 1,7$.



If Newton's method is used to determine an x -intercept of f and $x_0 = -0,4$ is chosen as the initial value, to which x -intercept will Newton's method tend?

(A) $x = 2$

(B) $x = -1$

(C) $x = 1$

(D) no x -intercepts

1.14 The function $h(x) = \frac{3x^2 - 2x - 1}{x^2 + x - 2}$ has the following asymptotes:

(A) $y = 3$; $x = -2$; $x = 1$

(B) $y = \frac{1}{3}$; $x = -2$; $x = 1$

(C) $y = 3$; $x = -2$

(D) $y = \frac{1}{3}$; $x = -2$

1.15 $\int \frac{2}{\ln 4(7x+3)} dx =$

(A) $\frac{\ln(7x+3)^2}{7} + c$

(B) $\log_4(7x+3)^{\frac{2}{7}} + c$

(C) $\log_4(7x+3)^2 + c$

(D) $2\ln 4 \cdot \ln(7x+3) + c$

QUESTION 2 [20 MARKS]

- 2.1 The intensity of sound, measured in watt/m², is given by the formula $I = 10^{-12}(e^{0,1x})$, where x is the volume of the sound, measured in decibels.
- (a) If the volume of a sound at a concert is 110 decibels, calculate the intensity of the sound. Give the answer in scientific notation. (2)
- (b) Damage to the hearing occurs if the intensity of a sound is greater than $8,1 \times 10^{-9}$ watt/m². What is the maximum volume that a sound can be, before hearing damage occurs? (3)
- 2.2 Given $p = \sqrt{2} - \sqrt{2}i$ and $k = 2e^{\frac{\pi}{3}i}$
- (a) Write p and k in polar form. (3)
- (b) Determine $z = p.k$. (2)
- (c) Determine \sqrt{z} . (3)
- 2.3 Solve for x if $x \in \mathbb{R}$: $|3x - 1| = \frac{2}{x}$ (7)

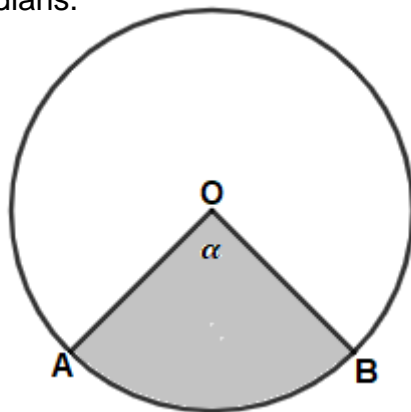
QUESTION 3 [23 MARKS]

- 3.1 In the binomial expansion of $(px + \frac{1}{x^2})^6$ the constant term will be equal to 240. Determine the value(s) of p . (8)
- 3.2 Use mathematical induction to prove that the following statement is true for all $n \in \mathbb{N}$:

$$\sum_{r=1}^n 3^{r-1} = \frac{1}{2}(3^n - 1)$$

(9)

- 3.3 The sketch below shows a circle with center O and radii AO en BO . $\widehat{AOB} = \alpha$ radians.



- (a) Show that $\alpha = \frac{\pi}{4}$ radians, if the area of the circle is eight (8) times greater than the area of sector AOB . (3)
- (b) Determine the perimeter of the shaded sector AOB if $AO = 3\text{cm}$. Give your answer in terms of π . (3)

QUESTION 4 [24 MARKS]

- 4.1 Given: $f(x) = 2\arcsin\left(x + \frac{3}{2}\right)$.
- (a) Show that f has no y -intercept. (2)
- (b) The domain of f is $x \in \left[-\frac{5}{2}; -\frac{1}{2}\right]$. Determine the range of f . (5)
- 4.2 Vector \mathbf{u} has a magnitude of 2 cm in a direction $\frac{\pi}{6}$ radians north of east. Calculate the horizontal and vertical components of \mathbf{u} . (2)
- 4.3 Determine the magnitude of vector \mathbf{a} if $\mathbf{a} \cdot \mathbf{b} = 10$ and vector \mathbf{b} has a magnitude of 4 cm. The angle between \mathbf{a} and \mathbf{b} is $\frac{\pi}{3}$ radians. (3)
- 4.4 The points $A(1; -1; 1)$, $B(2; 1; 3)$ and $D(-2; 1; 1)$ are three points, on the same level, in a three dimensional space.
- (a) Determine the vectors \mathbf{AB} and \mathbf{AD} . (2)
- (b) Determine the unit vector of \mathbf{AB} . (2)
- (c) Determine the angle between \mathbf{AB} and the y -axis. (3)
- (d) Are \mathbf{AB} and \mathbf{AD} parallel? Use the cross product and motivate your answer. (5)

QUESTION 5 [25 MARKS]

- 5.1 Given: $f(x) = \begin{cases} 2k + 1 & \text{if } x < 3 \\ t & \text{if } x = 3 \\ \frac{x}{k} & \text{if } x > 3 \end{cases}$
- (a) Determine the value(s) of k if f is continuous at $x = 3$. (5)
- (b) If $k = -\frac{3}{2}$, determine if f is differentiable at $x = 3$. (3)
- 5.2 Sketch a possible representation of a function f' (the first derivative of f) if the following is true:
- The function f has stationary points at $x = -1$ and $x = 2$.
 - The function f has inflection points at $x = 0$ and $x = 2$.
 - $f''(x) > 0$ if $x < 0$ and $x > 2$
 - f bends concave down if $0 < x < 2$ (7)
- 5.3 Given: $y = \ln\left(\frac{1}{xy}\right)$
- (a) Use implicit differentiation to determine $\frac{dy}{dx}$.
(Hint: First write the function as the difference between two logarithmic functions.) (7)
- (b) Hence, or otherwise, determine the gradient of the tangent line of the function at the point $\left(\frac{1}{e}; 1\right)$. (3)

QUESTION 6 [24 MARKS]

- 6.1 Show algebraically that $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$, $x \neq 0$, has a minimum turning point at (1; 2). (11)
- 6.2 (a) Determine the x -coordinate of the inflection point(s) of the function $h(x) = e^{\arctan x}$. (10)
- (b) Describe **in words** how it can be tested that these points are indeed inflection points. (It is not necessary to perform the test.) (3)

QUESTION 7 [22 MARKS]

- 7.1 Differentiate the following functions, as asked:
- (a) Determine $f'(x)$ if $f(x) = 2^{\cos^2(3x)}$ (5)
- (b) $D_x[\log_5(2x - 1) - \arcsin(\sqrt[3]{e^{-x}})]$ (6)
- 7.2 Use a Riemann sum to calculate the value of $\int_1^3 -2x^2 dx$ (11)

QUESTION 8 [22 MARKS]

- 8.1 Simplify:
- (a) $\int \left(5^{3x} - \frac{x^3}{5}\right) dx$ (5)
- (b) $\int \frac{dx}{x(\ln x)^3}$ (Hint: You may use substitution.) (4)
- (c) $\int \left(\cot^2(x) + \sec^2\left(\frac{x}{3}\right)\right) dx$ (5)
- 8.2 If $g(x) = \frac{2x^2 - 2x + 5}{(x-1)^2(4+x^2)} = \frac{A}{(x-1)^2} + \frac{1}{4+x^2}$, determine $\int g(x) dx$ (8)

QUESTION 9 [10 MARKS]

- Use integration by parts and show that the area below $y = (x + 1)e^x$, between $x = 0$ and $x = p$, can be written as pe^p . (10)

Total: 200**END OF PAPER**