α-MATHEMATICS

Alpha Mathematics
END EXAM PAPER

October 24, 2022 **Time: 3 hours**

Grade 12 **Total: 200 marks**

INSTRUCTIONS AND INFORMATION

Carefully read through the following instructions before answering the examination paper:

1. Answer all 10 questions on this examination paper.
2. Write your name and ID number on the front page of the examination paper.
3. Non-programmable calculators may be used, unless otherwise indicated at a specific question.
4. Unless indicated otherwise, all answers, where applicable, must be given correct to two decimal places.
5. The diagrams in the examination paper are not necessarily drawn to scale.
6. All angles are given in radians. Answers must be given in radians where applicable.
7. This examination paper consists of a front page, 24 pages and a formula sheet of 3 pages.
8. Question 1 consists of 10 multiple choice questions. Answer it on the answer sheet. This answer sheet is at the front of the paper. **Do not remove the answer sheet from the examination paper**.
9. For all other questions, all necessary calculations must be shown clearly. The correct answer on its own will not necessarily lead to full marks.
10. Additional writing space is provided at the end of this examination paper. Clearly indicate if you made use of this to complete a question.
11. Write neatly and legibly.

QUESTION 1 **[20 MARKS]**

- Answer this question **on the answer sheet** that is attached to the front, by marking A, B, C or D with an X (cross). Each question counts 2 marks.
- Please **DO NOT** remove this page from the examination paper.

1.1 When tested if a function f is continuous in the point $x = a$ and there is found that $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, then

(A) there exists a jump discontinuity at $x = a$.

(B) there exists a removable discontinuity at $x = a$.

(C) the function is differentiable in the point $x = a$.

(D) the function is continuous in the point $x = a$.

1.2 The binomial expansion of $\left(2x^2 + \frac{1}{x}\right)^{12}$ has a term px^3. What value must be used in the binomial formula for $\boldsymbol{r}$ to determine the value of p?

(A) 6 **(B)** 7

(C) 8 **(D)** 9

1.3 What does $f(x_i)$ represent in the following formula for a Riemann-sum?

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i)$$

(A) Width of the i-th rectangle **(B)** Number of rectangles

(C) The i-th rectangle **(D)** Hight of the i-th rectangle

1.4 Which statement is always true:

(A) If the gradients of tangents to a graph decrease as x increases, then the graph is concave up.

(B) If $f'(x) > 0$ then the graph will lie above of the x-axis.

(C) If $f'(b) = 0$ then there is a stationary point at $x = b$.

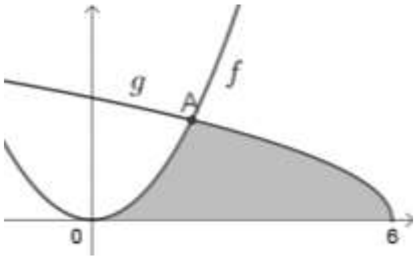
(D) If $f"(b) = 0$ then there is a point of inflection at $x = b$.

1.5 If $-\frac{2}{|x-1|} = 1,$ then

(A) $x = 3$ or $x = -1$ **(B)** there is no solution

(C) $\boldsymbol{x} \in \mathbb{R}$ **(D)** $x = 1$

1.6 The sketch shows the graph of f and g that intersect at the point A(2; 2). The respective x-intercepts of f and g are (0; 0) and (6; 0).



Which expression can be used to determine the area of the shaded region?

(A) $\int_0^6 (f+g)\,dx$ **(B)** $\int_0^6 g\,dx$

(C) $\int_0^6 f\,dx - \int_2^6 g\,dx$ **(D)** $\int_0^2 f\,dx + \int_2^6 g\,dx$

1.7 $\int_0^a \sin^2 2x\,dx$

(A) $\frac{\sin^3 2a}{6}$ **(B)** $\frac{a}{2} - \cos 2a$

(C) $\frac{a}{2} - \frac{\sin 4a}{8}$ **(D)** $\frac{a}{2} - \frac{\sin 2a}{4}$

1.8 The inverse of $f(x) = \tan(2x - \frac{\pi}{4})$ will be defined if the domain of f is restricted to $x \in$

(A) $(-\frac{\pi}{2}; \frac{\pi}{2})$ **(B)** $(-\frac{\pi}{8}; \frac{3\pi}{8})$

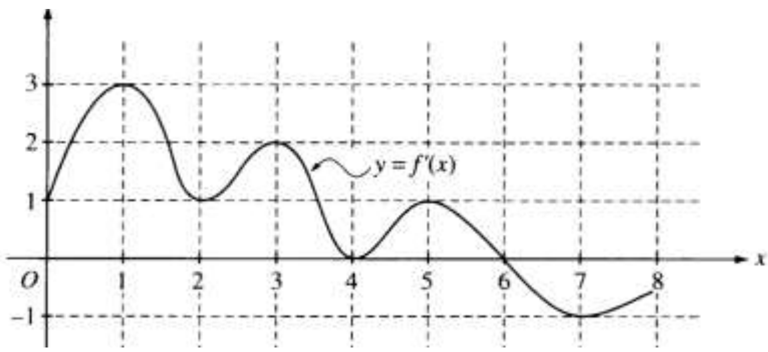
(C) $(-\infty; \frac{3\pi}{8})$ **(D)** $(-\frac{\pi}{4}; \frac{3\pi}{4})$

1.9 If $e^y = sinx, 0 < x < \frac{\pi}{2}$, then $\frac{dy}{dx} =$

(A) $\frac{1}{\tan x}$ **(B)** $\frac{-1}{\tan x}$

(C) $\tan x$ **(D)** $-\tan x$

1.10 The sketch shows the graph of $y = f'(x)$, the derivative of f.The point (3; 5) lies on the graph of f. The equation of the tangent to f at the point (3;5) is:



(A) $y = 2$ **(B)** $y = 3x - 4$

(C) $y = 2x - 1$ **(D)** $y = 2x + 4$

Answer the following questions **on the exam paper** on the lines provided at the end of each question. Clearly indicate if you use the additional writing space at the end of the paper to complete a question.

QUESTION 2 **[20 MARKS]**

2.1 The world population can currently be determined with the formula $\boldsymbol{P = 6,9(1,011)^t}$, where P is the population in billions and t the number of years after 2011.

(a) Determine the size of the population in 2050 in billions. (3)

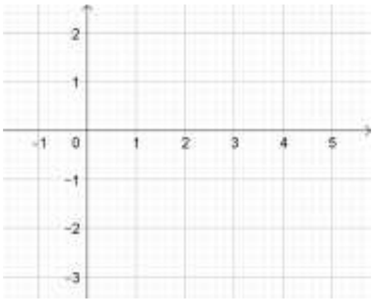
(b) In which year will the population double from the size in 2011? (4)

(c) Determine the tempo at which the population is growing in 2022. Give the answer to the nearest million per year. (Accept 1 billion = 10^9) (5)

2.2 Given $\boldsymbol{f(x) = \ln(x+1) - 1}$.

(a) Determine the intercepts of the graph of $\boldsymbol{f}$ with the axes. (4)

(b) Sketch a graph of the function on the following set of axes. Clearly show the asymptote and the intercepts with the axes. (4)



QUESTION 3 **[23 MARKS]**

3.1 Solve for x: $\boldsymbol{|4x - 12| = -6x + 6}$ (6)

3.2 (a) Factorise $\boldsymbol{x^4 - 4}$ fully in $\mathbb{C}$**[x].** (4)

(b) The equation $\boldsymbol{2x^5 - 13x^4 + 30x^3 - 36x^2 + 20x - 4 = 0}$ has two irrational roots, two non-real roots and one rational root.
As $\boldsymbol{1 + i}$ and $\boldsymbol{2 - \sqrt{2}}$ are two of the roots, calculate the rational root. (8)

3.3 Give the first 4 simplified terms of the following power series: $\boldsymbol{\sqrt[3]{1-3x}}$. (5)

QUESTION 4 **[20 MARKS]**

4.1 Use mathematical induction and prove that (10)

$$\sum_{p=1}^{n}(3p-2)^2 = \frac{1}{2}n(6n^2 - 3n - 1)$$

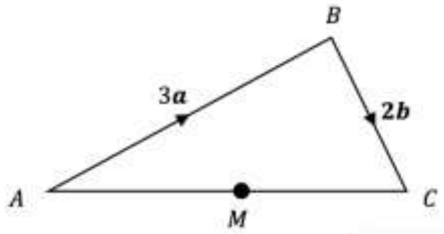
4.2 Calculate the value of $\left(-\sqrt{3}+i\right)^{8}$. Use de Moivre's theorem and do the calculation in polar form. The answer must be given in rectangular form. All steps must be in terms of π and root form where necessary. (7)

4.3 Show how Cramer's rule can be used to solve for x in the following set of equations. Do not calculate the determinants, only show how x can be solved by reference to the applicable matrices. (3)

$$\begin{aligned} 5x + 2y - z &= 7 \\ y - 4z &= 9 \\ 9x + \quad 4z &= 2 \end{aligned}$$

QUESTION 5 **[18 MARKS]**

5.1 The sketch shows triangle ABC with vectors $\boldsymbol{AB} = 3\boldsymbol{a}$ and $\boldsymbol{BC} = 2\boldsymbol{b}$. The point $\boldsymbol{M}$ is the midpoint of $\boldsymbol{AC}$. Determine an expression for vector $\boldsymbol{AM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. (3)



5.2 D(2; -1; 4), E(-1; 0; 2) and F(4; 5; -1) are three points in a three dimensional space.

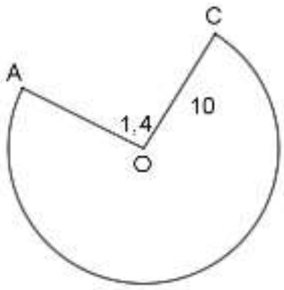
(a) Determine vectors $\boldsymbol{u} = \boldsymbol{DE}$ and $\boldsymbol{v} = \boldsymbol{EF}$ in the form $\boldsymbol{ai} + \boldsymbol{bj} + \boldsymbol{ck}$. (2)

(b) Use the dot product and determine the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. (6)

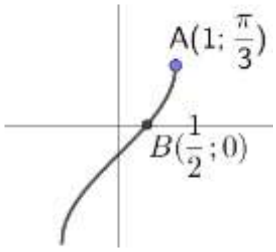
(c) Use the cross product and determine the area of the parallelogram that is formed by these two vectors as sides. (7)

QUESTION 6 **[23 MARKS]**

6.1 A vertical grain silo was erected on the corner of a farmer's barn with the ground plan like the sketch. The nearly ¾ circle has a radius of 10 m, $A\hat{O}C = 1{,}4$ radians and the silo is 20 m high. The silo is 80% full. Grain weighs 760kg/m^3. Calculate how many kg of grain the silo contains. (4)



6.2 The sketch shows the graph of $f(x) = a.\arccos(x) + p$. The points $A(1; \frac{\pi}{3})$ and $B(\frac{1}{2}; 0)$ lie on the graph. Calculate the values of a and p. (4)



6.3 $g(x) = \begin{cases} bx - 8 & \text{if} \quad x \leq b \\ 2b & \text{if} \quad b < x \leq 1 \\ -4(x-1)^2 + c & \text{if} \quad x > 1 \end{cases}$

(a) The function g is continuous for all $x \in \mathbb{R}$. Calculate the values of b and c. (5)

(b) Motivate algebraically if g is differentiable at $x = 1$. (4)

6.4 If $f(x) = (\ln x)^2 + x$ determine the x value of the point on f where the tangent to f will have a **maximum gradient**. (6)

QUESTION 7 **[20 MARKS]**

7.1 Differentiate the following functions as specified:

(a) Determine $\frac{dy}{dx}$ if $y = e^{\tan 3x} - \frac{4}{x}$. (4)

(b) $D_x\left[\arcsin(\sqrt{1 - x^2})\right]$ and simplify the answer if $x > 0$. (5)

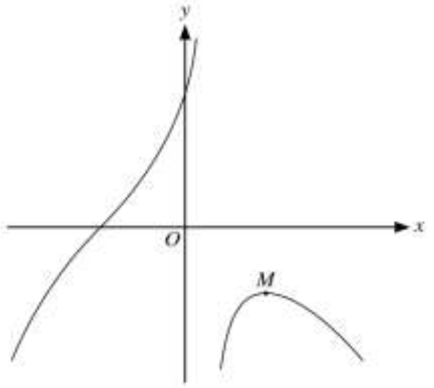
(c) Determine $f'(x)$ if $f(x) = \log(\cos x)$. (3)

7.2 Determine the equation of the normal to the given function at the point (3; 1).
Use implicit differentiation.
(The normal is perpendicular to the tangent at the point.)

$$x^2 \ln y + 2x + 5y = 11$$ (8)

QUESTION 8 **[20 MARKS]**

8.1 The sketch shows the graph of $\boldsymbol{y = \frac{x^3+8}{2-5x}}$:



(a) Show that the $\boldsymbol{x}$-value of the turning point M can be calculated with the equation $\boldsymbol{5x^3 - 3x^2 - 20 = 0}$. (4)

(b) Use Newton's method and calculate this $\boldsymbol{x}$-value, correct too three decimal digits. Use $\boldsymbol{x = 2}$ as first approximation. Clearly show how you use Newton's method. (4)

8.2 The following information is available about a continuous function $\boldsymbol{y = h(x)}$ of degree 4.

$$\boldsymbol{h(0) = -3} \text{ and } \boldsymbol{h(3) = 0}$$

$$\boldsymbol{h'(0) = 0} \text{ and } \boldsymbol{h'(3) = 0}$$

$$\boldsymbol{h''(0)=0;} \text{ and } \boldsymbol{h''(3) = -4}$$

(a) Give the function's intercept(s) with the x-axis. (1)

(b) Give the x-values where the function has stationary points.
Also state **with reasons** which type of stationary point each one is. (6)

8.3 Name the type of asymptotes of f and determine the equations of all these asymptotes if: $\boldsymbol{f(x) = \frac{(x-2)(x^2+x-1)}{x^2-4}}$. (5)

QUESTION 9 **[20 MARKS]**

9.1 Use a Riemann sum and determine $\boldsymbol{\int_0^3 (-x^2 + 1)\, dx}$. (9)

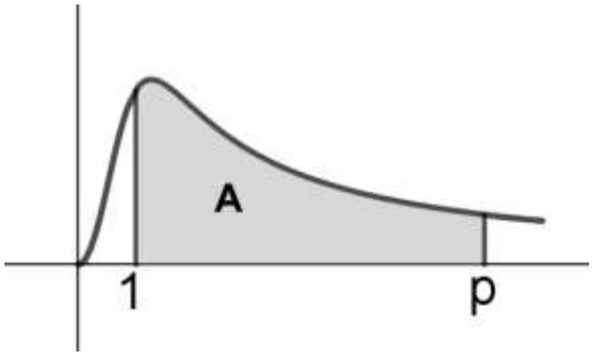
9.2 Determine $\boldsymbol{\int (\sin(5x) \times \cos(2x))\, dx}$. (4)

9.3 Use partial fractions and determine $\boldsymbol{\int \frac{x+1}{(x-2)^2}\, dx}$. (7)

QUESTION 10 **[16 MARKS]**

10.1 The graph of $\boldsymbol{y = \sqrt{x}.e^{x}}$ rotates around the x-axis. Determine the volume of the body of revolution that is formed between the lines $x = 0$ and $x = 2$.
Give the answer in terms of $\boldsymbol{e}$ and $\boldsymbol{\pi}$.
Show all steps used in the partial integration. (8)

10.2



The sketch shows the graph $\boldsymbol{f(x) = \frac{x^2}{1+x^3}}$ for $x \geq 0$. The shaded region A is the region included by the graph, the x-axis and the lines $x = 1$ and $x = p$.
The value of p is equal to $\boldsymbol{\sqrt[3]{a.e^{c} + b}}$ when the area of A is equal to 2.
Determine the values van $\boldsymbol{a, b}$ and $\boldsymbol{c}$. (8)

TOTAL: 200 MARKS

