AUTHOR: RIKA GROBLER………………………………

ISBN 1 919740-19-8

Content

MARKS DISTRIBUITION FOR END OF YEAR EXAMINATION

NOTE: Question 1 consists of 15 multiple choice questions and counts 30 marks. Questions will not need more than one to two minutes.

Algebra: ±65

- Absolute Value
- Partial Fractions
- Solve polynomial equations
- The Binomial Theorem
- Power series
- Mathematical Induction
- The natural logarithm and exponent
- Cramer’s rule
- Complex numbers, operations in different forms, De Moivre’s theorem
- Roots of complex numbers

Trigonometry: ±15

- Inverse trigonometric functions, with graphs and transformations
- Application of radian measure in sectors

Vectors: ±15

Calculus: ±105

Differentiation:

- Limits, continuity and differentiability
- Use of differentiation rules
- Differentiate exponents and logarithms. Also e^x and natural logarithms.
- Implicit differentiation
- Higher order derivatives and the meaning thereof.
- Newton’s method
- Optimization
- Sketch of rational functions

Integration:

- Use table, also for linear functions
- Trigonometric integration by using identities
- Factor integration (integration by parts)
- Integration by means of partial fractions
- Integration where substitution can be used
- Area under and between graphs
- Volume of rotating bodies
- The Riemann sum

NOTE: The logarithm rules are always provided on the information sheet. It can be expected of learners to use these.

ALGEBRA

ABSOLUTE VALUE

Absolute value is written with two vertical lines: $|x|$.

Absolute value is ALWAYS positive. It makes negative numbers positive. Positive numbers remain unchanged.

There are 4 types of problems:

1. Equations with a number outside the absolute value.
2. Equations with x outside the absolute value.
3. Inequalities.
4. Graphs.

Here follows examples of these four types of problems.

Type 1. These are equations with absolute value and only a number outside the absolute value. It is mostly used to calculate the x-intercepts of the graphs.

Example 1

1 Solve for x:

1.1 $|x-2|=-4$

It is impossible, because absolute value cannot be equal to a negative number. Therefore no answer.

1.2 $|x-2|=4$

When the absolute value symbol is left out, whatever is written inside can be either positive or negative. Thus:

$x-2=4$ OR $-x+2=4$. Normally we don't write this step, but continue to the next step: $x-2=4$ or $x-2=-4$. Thereafter the answers: $x=6$ or $x=-2$

Type 2. These are equations with absolute values with an unknown outside the absolute value. With these problems there always need to be two calculations and each one must have a condition at the front.

Example 2

2.1 Solve for x: $|\boldsymbol{x-4}| = \mathbf{2} - \mathbf{3}\boldsymbol{x}$

This problem has an x outside the absolute value. It needs to be done in two steps and the condition of each step must be written down. If you don't do this, no marks will be awarded.

What is the condition?

The part inside the absolute value can be positive (≥ 0) or negative (< 0).

Why is the "equal to" with the "bigger than"? Because mathematicians decided that, therefore the definition.

Always start these equations with the conditions, write them next to each other as shown. (You already receive 2 marks by doing so):

$\boldsymbol{x-4} \geq 0$ OR $\boldsymbol{x-4} < 0$

Get x alone:

$\boldsymbol{x \geq 4}$ OR $\boldsymbol{x < 4}$

Remember: With the first condition, everything stays the same; with the second condition, the sign has to change, but ONLY whatever is inside the absolute value. Now there are two equations that need to be solved. See if the answer falls within the restriction.

This type of problem can have none, one, or two solutions. Here is a complete solution:

Conditions:

$\boldsymbol{x \geq 4}$	OR $\boldsymbol{x < 4}$
Then	Then
$x - 4 = 2 - 3x$	$-x + 4 = 2 - 3x$
$\therefore 4x = 6; \ \therefore x = \frac{6}{4} = \frac{3}{2}$	$\therefore 2x = -2; \ \therefore x = -1$
Now check: Is $\frac{3}{2} \geq 4$?	Now check: Is $-1 < 4$
NO, thus no answer.	And the answer is **YES**

Therefore the problem only has one solution: $\boldsymbol{x = -1}$**.**

The graphs of $y = |x - 4|$and $y = 2 - 3x$ will only intersect in one place.

2.2 Solve for x: $|\boldsymbol{x+4}| = \boldsymbol{x}$

Solution

Conditions:

$\boldsymbol{x+4 \geq 0, \therefore x \geq -4}$ OR $\boldsymbol{x+4 < 0; \therefore x < -4}$

(Remember: 2 marks!)

Then

$x + 4 = x$, no solution OR $-x - 4 = x;$

$\therefore -2x = 4; \ \therefore x = -2$

Is $-2 < -4$?

NO

thus no solution.

This problem has **NO** solution.

The graphs of $y =: \ |x + 4|$ and $y = x$ will never intersect.

Type 3.

These are inequalities with absolute value, but only with numbers outside the absolute value.

3 Inequalities with smaller than (<) or bigger than (>).

The < is the easiest, because the absolute value must be between the negative and the positive number.

Example 3

3.1 Solve for x: $|\boldsymbol{x-4}| < \mathbf{7}$

Solution

Write as: $-7 < \boldsymbol{x-4} < 7$

With answer: $-3 < x < 11$

The > has more work, it must be written as OR, because it is smaller than the negative OR bigger than the positive.

3.2 Solve for x: $|\boldsymbol{x-4}| > \mathbf{7}$

Solution

Then $\boldsymbol{x-4} < \mathbf{-7}$ **OR** $\boldsymbol{x-4} > \mathbf{7}$

With answer: $x < -3$ OR $x > 11$

Note that the answer is exactly the opposite of 3.1.

What happens if there is also a "="? It is done in exactly the same way:

3.3 Solve for x: $|\boldsymbol{x-4}| \leq \mathbf{7}$

Write as: $-7 \leq \boldsymbol{x-4} \leq 7$

With answer: $-3 \leq x \leq 11$

Three exceptions:

3.4 Solve for x: $|\mathbf{x-4}| < \mathbf{-7}$

It is impossible, because absolute value is always positive. It can therefore not be smaller than a negative number. Therefore no solution.

3.5 Solve for x: $|\mathbf{x-4}| > \mathbf{-7}$

This is always true, because absolute value is positive and therefore always greater than a negative number. Answer therefore $\boldsymbol{x} \in \mathbb{R}$**.**

3.5 Solve for x: $|\mathbf{x-4}| \leq \mathbf{0}$

Absolute value cannot be negative or less than zero, but here is also a = sign. Absolute value can be equal to 0 (zero).
Therefore the answer is: $\mathbf{x = 4}$.

If these problems are asked with graphs, you should remember that on any graph the greater than (>) is above the x-asis, and the less than (<) below the x-asis. This will be explained later at graphs.

Type 4.

Graphs
The question can be asked in 3 ways:

1 Draw a graph of a linear absolute value.

2 The graph is given, and you have to answer questions on it.

3 Another type of graph is given, and you have to draw the absolute value of this graph.

Example 4
4.1 With these questions the intercepts with the axes, the vertex and the shape must be determined.

Sketch $\boldsymbol{y = |x - 1| - 2}$

Solution
x-intercepts, let $y = 0$. Now it is as in e.g. 1.1.

$|x - 1| - 2 = 0; \therefore |x - 1| = 2$

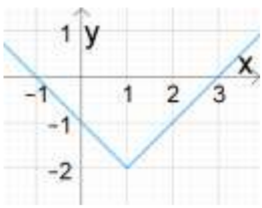
Thus $x - 1 = \pm 2$; $\boldsymbol{x = 3}$ or $\boldsymbol{x = -1}$

y-intercept, let $x = 0$: $\boldsymbol{y} = |0 - 1| - 2 = |-1| - 2 = \boldsymbol{-1}$

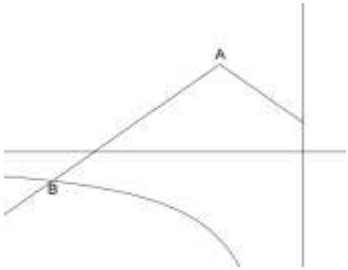
Vertex: (Remember, x opposite sign) $(1; -2)$

Shape: V, because absolute value is positive.

Sketch:



4.2 The following sketch shows the graphs of $f(x) = -|x+2| + 3$ and $g(x) = \frac{6}{x}$ for $x \leq 0$



(a) Give the coordinates of A, the vertex of $f(x)$.

(b) Calculate the coordinates of B, the intersection of $f(x)$ and $g(x)$.

(c) Use your answer in (b) along with the graph and give the x-values where $f(x) > g(x)$.

Solution:

(a) $(-2; 3)$

(b) B lies on +(positive) gradient, $\therefore y = x + 2 + 3$;

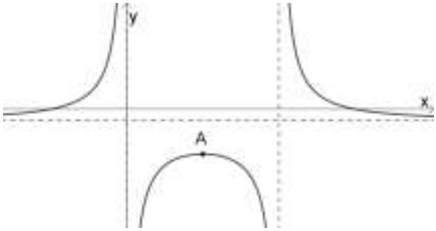
$$y = x + 5$$

Hence $x + 5 = \frac{6}{x}$; $x^2 + 5x - 6 = 0$; $x = -6$; $\therefore y = -1$

(c) $f(x) > g(x)$ means f above g, but not included, because not = :

Thus, for $-6 < x < 0$

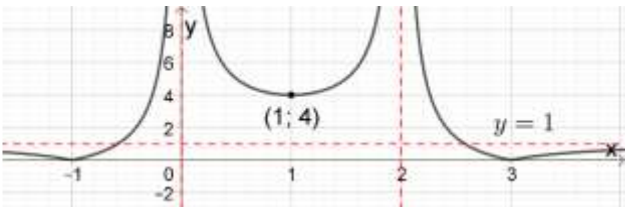
4.3 The sketch shows the graph of $f(x) = \frac{x^2-2x-3}{2x-x^2}$.



Make a neat sketch graph of $g(x) = \left|\frac{x^2-2x-3}{2x-x^2}\right|$.

Solution:

Everything below the x-axis must reflect in the axis, including the asymptote which is currently negative. In the original question the vertex and intercepts need to be calculated. That is not applicable now, but is shown on the sketch:



PARTIAL FRACTIONS

This question is usually combined with integration.
There are three types you should be able to do.

1. Only linear factors in the denominator that do not repeat.
2. Only linear factors in the denominator that repeat.
3. Quadratic factors in denominator that cannot be factorised in the real number system.

There are steps that are the same:

- Ensure the degree of the denominator is bigger than the degree of the numerator, otherwise first do long division.
- Make sure that the denominator has been factorised completely in $\mathbb{R}$.
- Write the fraction as separate fractions. (will show in example)
- Multiply by the LCM.
- If there are linear factors, substitute their zeros in x.
- Compare the coefficient of the LHS and the RHS. Always start with the highest exponent of x.
- Write down the answer.

Type 1

Example 5

Decompose $\frac{x+7}{(x-2)(x+1)}$ into partial fractions.

Solution

This fraction started by adding two fractions, of which the denominators were $x-2$ and $x+1$

Now set: $\frac{x+7}{(x-2)(x+1)} \equiv \frac{A}{x-2} + \frac{B}{x+1}$

Multiply with the **LCM,** $(x-2)(x+1)$**:**

$$x+7 \equiv A(x+1) + B(x-2)$$

Zero of $(x-2)$, let $x=2$:
$\therefore 2 + 7 = A(2+1) + B(2-2)$; $9 = 3A$,$\therefore$ $A=3$
Zero of $(x+1)$, let $x=-1$:
$\therefore -1+7 = A(-1+1) + B(-1-2)$, $6 = -3B$, $\therefore$ $B=-2$
Substitute the $\mathbf{A}=\mathbf{3}$ and $\mathbf{B}=-\mathbf{2}$:

$$\therefore \frac{x+7}{(x+1)(x-2)} \equiv \frac{\mathbf{3}}{x-2} - \frac{\mathbf{2}}{x+1}$$

Type 2

Example 6

Decompose $\frac{2x^2+x-15}{(x^2-2x+1)(x+2)}$ into partial fractions.

Solution

Take note that the denominator doesn't factorise completely:

$$\frac{2x^2+x-15}{(x-1)^2(x+2)}$$

The denominator is: $(x-1)$, $(x-1)^2$ and $(x+2)$

Linear factors that repeat

Now set:

$$\therefore \frac{2x^2+x-15}{(x-1)^2(x+2)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

Multiply each fraction with the LCM: $(x-1)^2(x+2)$

$2x^2+x-15 \equiv A(x-1)(x+2) + B(x+2) + C(x-1)^2$

Let $x = 1$: $\therefore -12 = A(0)(3) + B(3) + C(0),$
$-12 = 3B,\ B = -4$

Let $x = -2$ $\therefore -9 = A(-3)(0) + B(0) + C(-3)^2$

$-9 = 9C,\ C = -1$

Now we need A, but there aren't any zeros. Look at the identity at the top, the highest exponent of x is 2. And the LHS = RHS.
On the left there is 2 and on the right there is $A + C$. Thus

$$2 = A + C, \qquad \therefore 2 = A - 1, \quad \therefore A = 3$$

$$\therefore \frac{2x^2+x-15}{(x-1)^2(x+2)} \equiv \frac{3}{x-1} - \frac{4}{(x-1)^2} - \frac{1}{x+2}$$

Type 3

Example 7

Decompose $\frac{5x^2-4x+3}{(\boldsymbol{x-1})(\boldsymbol{x^2+1})}$ into partial fractions.

Solution

Take note that $x^2 + 1$ cannot be factorised further.

Now set:

$$\frac{5x^2 - 4x + 3}{(x-1)(x^2+1)} \equiv \frac{\mathrm{A}}{\boldsymbol{x-1}} + \frac{\mathrm{B}x + \mathrm{C}}{\boldsymbol{x^2+1}}$$

Multiply with the **LCM**: $(x-1)(x^2+1)$

$$\therefore 5x^2 - 4x + 3 \equiv \mathrm{A}(x^2+1) + (\mathrm{B}x + \mathrm{C})(x-1)$$

Always try the zeros of the denominator first:

Let $\boldsymbol{x = 1}$: $\quad \therefore 4 = 2\mathrm{A}$, $\mathbf{A} = \mathbf{2}$

The highest exponent is 2. State LHS = RHS:

$5 = A + B$, $\therefore 5 = 2 + B$, Thus $\boldsymbol{B} = \mathbf{3}$.

Name the constants and state LHS = RHS:

$3 = A - C$, $\therefore 3 = 2 - C,$ Thus $\boldsymbol{C} = \mathbf{-1}$

$$\therefore \frac{5x^2 - 4x + 3}{(x-1)(x^2+1)} = \frac{2}{x-1} + \frac{3x-1}{x^2+1}$$

SOLVE POLYNOMIAL EQUATIONS

It is important to read the question correctly. You must:

- Factorise
- Solve the equation, where you also have to factorise.

You also have to read carefully how the answer has to be: $\mathbb{Q}$ which is rational, $\mathbb{R}$ which is real, $\mathbb{C}$ which is complex

Remember the following:

- Rational zeros will always be $\frac{a_0}{a_n}$, where a_0 is the constant and a_n is the higher coefficient.
- Irrational and complex zeros always occur in pairs. Be careful, the irrational or imaginary part gets the $\pm$. Question papers sometimes write the imaginary part first, e.g. $2i - 4$. The other zero is $-2i - 4$.

Example 8

1 Factorise $x^4 + 5x^3 + 27x^2 + 5x - 174$ if $x = \mathbf{-2 - 5}\boldsymbol{i}$ is a zero. Give the answer in $\mathbb{Q}$.

Solution (first we are going to get the quadratic factor. There are various ways, but this one is the easiest.)

$x = -2 \pm 5i$, thus $x + 2 = \pm 5i$

$(x + 2)^2 = (\pm 5i)^2$ ... square

$x^2 + 4x + 4 = 25i^2$ ...$i^2 = -1$

Thus $\boldsymbol{x^2 + 4x + 29}$ is a factor of the polynomial.

Now the other quadratic factor has to be determined. You can use long division or inspection.

$(\boldsymbol{x^2 + 4x + 29})(\boldsymbol{x^2 + x - 6})$

$= (\boldsymbol{x^2 + 4x + 29})(\boldsymbol{x - 2})(\boldsymbol{x + 3})$

These are the factors with rational zeros as asked.

Example 9

Solve for x: $x^4 + 5x^3 + 27x^2 + 5x - 174 = 0$ if $x = \mathbf{-2 - 5i}$ is a root. Give the answer in $\mathbb{C}$.

Solution

Exactly the same steps as in the previous problem need to be done, except that there now is an equation. A few more steps need to be done. The answer has to be complex, which means that the factorisation must be complex:

$$(\mathbf{x^2 + 4x + 29})(\mathbf{x - 2})(\mathbf{x + 3}) = \mathbf{0}$$

$$(\mathbf{x + 2 + 5i})(\mathbf{x + 2 - 5i})(\mathbf{x - 2})(\mathbf{x + 3}) = 0$$

$\therefore x = -2 \pm 5i$ or $x = 2$ or $x = -3$

With this problem the last step of factorisation is unnecessary. The Quadratic polynomial is determined by the roots. You could have gone directly to the answer after first factorisation.

THE BINOMIAL THEOREM

The formula used here appears on the formula sheet:

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

This formula can only be used if n is a positive integer. There are basically three types of questions that can be asked.

Type 1.

The question is to expand a binomial. It will not be asked for powers higher than 4:

Example 10
Use the binomial theorem and expand the following:
$(\boldsymbol{x} + \boldsymbol{2y})^{\boldsymbol{4}}$

Solution

$$(\boldsymbol{x} + \boldsymbol{2y})^{\boldsymbol{4}} = \binom{\boldsymbol{4}}{0}\boldsymbol{x}^4(\boldsymbol{2y})^0 + \binom{\boldsymbol{4}}{1}\boldsymbol{x}^3(\boldsymbol{2y})^1 + \binom{\boldsymbol{4}}{2}\boldsymbol{x}^2(\boldsymbol{2y})^2 + \binom{\boldsymbol{4}}{3}\boldsymbol{x}^1(\boldsymbol{2y})^3 + \binom{\boldsymbol{4}}{4}\boldsymbol{x}^0(\boldsymbol{2y})^4$$

$$= x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$$

Type 2

At these questions a specific term is asked:

REMEMBER the r'th term is in the $(r-1)$'th position, in other words r is one less than the position of the number

Example 11

Determine the **6** th term of $\left(2x - \frac{3}{x^2}\right)^{10}$.

Solution

Always write down the values of a, b, n and r:

$a = 2x$, $b = -\frac{3}{x^2}$, $n = 10$, $r = 5$ $(6-1)$

Therefore $\binom{n}{r} a^{n-r}\, b^r = \binom{10}{5}\,(2x)^{10-5}\left(-\frac{3}{x^2}\right)^5$

First calculate the numbers with your calculator, and then do the $x's$.

$= -\frac{1959552}{x^5}$

Type 3

The question here is to give the coefficient of a term:

Example 12

Determine the coefficient of x^2 in the expansion of $\left(3x - \frac{4}{x}\right)^8$

Solution

The big problem here is to determine the value of r.

First write it down like this:

$$\binom{n}{r} a^{n-r}\, b^r = \binom{8}{r}\,(3x)^{8-r}\left(-\frac{4}{x}\right)^r$$

Look at the x's: $x^{8-r} \div x^r = x^2$

Thus $8 - r - r = 2$, it gives $r = 3$.

Thus $\binom{8}{3}\, 3^{8-3}\,(-4)^3\, x^2$

$= -870912x^2$

Therefor the coefficient of x^2 is $-$**870912**.

POWER SERIES

The questions are on binomials, but here the value of n is not a positive integer. Examples include:

$\frac{2}{(1+x)^4} = 2(1+x)^{-4}$, $\sqrt{1-2x} = (1-2x)^{\frac{1}{2}}$

The formula is provided on the formula sheet:

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \cdots \ ; \textit{if } |x| < 1$$

Note that the LHS is 1+.... If there is another number, it must be taken out as a common factor. Further, the formula only has an x, but questions often have something else, such as $-2x$ in the example above. This means that everywhere there is x on the RHS of the formula, it has to be substituted by $(-2x)$.

Example 13

Give the first four terms of the expansion of $\frac{1}{2+x}$. Also give the values of x for which the expansion is valid.

Solution:

$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}$, make sure there is a 1.

$= \frac{1}{2}\left(1 + (-1)\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3 + \cdots\right)$

Note that the x in the RHS of the formula is substituted by $\frac{x}{2}$.

$$= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \cdots$$

This result is valid/ converge if $\left|\frac{x}{2}\right| < 1$,

$\therefore |x| < 2$

MATHEMATICAL INDUCTION

This is 'n method to prove that two expressions are equal to each other for all values of the unknown. There are 4 steps that need to be done, if it says **"Prove**" then the LHS and RHS should be done separately:

1. **Prove** it is true for $n = 1$.
2. Assume the statement is true for $n = k$.
3. Use nr 2 and **prove** the statement is true for $n = k + 1$.
4. Write down the conclusion.

At steps 1 and 3 you have to **prove**, therefore LHS and RHS have to be done separately. The word "**Assume**" needs to be in your proof.

Example 14

Prove that $2 + 4 + 6 + \cdots + 2n = n(n + 1)$ for all natural numbers n.

- On the LHS, n is the number of terms.
- On the RHS, n has to be replaced.

Solution

STEP 1: **Test for $n = 1$**:

LHS $= 2$ (the first term)
RHS $= 1(1 + 1) = 2$

$\therefore$ LHS = RHS and the statement is true for a natural number, $n = 1$

STEP 2: **Assume the statement is true** for a natural number k, thus for $n = k$

$$\mathbf{2 + 4 + 6 + \cdots + 2k = k(k + 1)} \quad (*)$$

This step will be used in the next step. The first k elements that are added together, can be replaced/ substituted by the RHS of this equation.

STEP 3: **Consider $n = k + 1$:**

RHS = $(k+1)(k+1+1)$

Substitute $n = k + 1$ in the original equation.

$= (k+1)(k+2)$

LHS = $2 + 4 + 6 + \cdots + 2k + 2(k+1)$

From STEP 2, add the $(k+1)$'th term

$= k(k+1) + 2(k+1)$**….from (*)**

$= (k+1)(k+2)$

= RHS

STEP 4: The statement is therefore true according to mathematical induction.

Sigma Notation

Since the LHS of these types of problems is often addition, the sigma notation can be used.

Example 15

Prove that $$\sum_{k=1}^{n} k^2 = \frac{n}{6}(n+1)(2n+1)$$

This means that

$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n}{6}(n+1)(2n+1)$.

Solution

STEP 1: **Test for $n = 1$:**

LHS = 1, RHS = $\frac{1}{6}(\mathbf{1}+1)(2(\mathbf{1})+1) = 1$

$\therefore$ LHS = RHS and the statement is true if $n = 1$

STEP 2: **Assume** the statement is true for a natural number k, thus for $n = k$

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k}{6}(k+1)(2k+1)$$

We are going to use this step in the next:

STEP 3: **Consider $n = k + 1$:**

$$\text{LHS} = 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2$$

$$= \frac{k}{6}(k+1)(2k+1) + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$\text{RHS} = \frac{k+1}{6}(k+1+1)(2(k+1)+1)$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6} = \text{LHS}$$

STEP 4: The statement is therefore true according to mathematical induction.

In these types of problems, the LHS is not necessarily addition. It can also be multiplication of terms. In the 2019 gr 12 exam the proof of de Moivre's theorem has been asked. Any formula that is true for $n \in N$ can be proved using mathematical induction.

THE NATURAL LOGARITHM AND EXPONENT

The number e was discovered in research on compound interest. Afterwards, various formulas originated to determine the value of e, an irrational number. From this, the following was concluded:

The derivative of the function $f(x) = e^x$ is always equal to the function itself. Therefore, also the integral of the function:

$\frac{d}{dx}(e^x) = e^x$ and $\int e^x dx = e^x + k$.

Any exponent has an inverse logarithm. Therefore also e^x. This logarithm is so important, it got a special name:

If $f(x) = e^x$ then $f^{-1}(x) = ln(x)$

It is often written as $ln|x|$ because logarithms are only defined for positive numbers.

Furthermore, we know that this special logarithm gives the area between a hyperbola and the x-asis:

$\int \frac{1}{x} dx = lnx + k$ or $\frac{d}{dx}(lnx) = \frac{1}{x}$.

By means of these formulas, formulas for the differentiation of normal exponents and logarithms could be determined:

$\frac{d}{dx}(a^x) = a^x.lna$ and $\frac{d}{dx}(\log_a x) = \frac{1}{x.lna}$.

Important to know how to convert in an equation from an exponent to a logarithm and vice versa:

If $10 = e^x$, then $x = ln10$.
If $10 = lnx$, then $x = e^{10}$.

Furthermore $ln(e^x) = x$ and $e^{lnx} = x$.

Example 16

1 Determine $f'(x)$ if $f(x) = e^{cosx}$

2 Determine the inverse function of $f(x) = e^{3x} +$

Solutions

1 Use the chain rule: $f'(x) = e^{cosx} \times (-sinx)$

2 Swop x and y:
$x = e^{3y} + 2,\ \therefore e^{3y} = x - 2,\ \therefore 3y = ln|x - 2|$

Thus $f^{-1}(x) = \frac{1}{3} ln|x - 2|$

All properties of logarithms also hold for ln. When we need to differentiate, it is much easier if the function is first expanded using the logarithm rules.

Example 17

Determine $f'(x)$ if $f(x) = ln\ (\frac{x^2}{x+1})$

Solution

First use the logarithm laws:

$f(x) = lnx^2 - ln(x + 1) = 2lnx - ln\ (x + 1)$.

Thus $f'(x) = \frac{2}{x} - \frac{1}{x+1}$

CRAMER'S RULE

There are a few methods to solve systems of linear equations. Cramer's rule provides a method to solve a system of linear equations simultaneously by using determinants of matrices. In this course, we will only go up to three equations with three unknowns. If you have IT as a subject, you are welcome to try to write an algorithm for more equations.

What is useful about Cramer's rule, is that you can only solve one of the variables. It also provides a quick method to test if a system has no solutions – will show a bit later.

Cramer work with determinants, it means that you have to know how to calculate a determinant.

Cramer's rule
Consider a system of linear equations with $\boldsymbol{n}$ variables and $\boldsymbol{n}$ equations, that is represented in matrix form as follows:

$$\mathbf{A}x = \boldsymbol{b}$$

where $\mathbf{A}$ is the **square matrix** of which the **determinant ≠ 0** and $\boldsymbol{b}$ is the **"answer".** Then any of the variables can be calculated by using the following formula:

$$x_i = \frac{\det \mathbf{A}_{\boldsymbol{i}}}{\det \mathbf{A}}, \qquad i = 1, 2, \dots n$$

where $\mathbf{A}_{\mathbf{i}}$ is the matrix obtained from substituting the $\boldsymbol{i}$**'th column** with $\boldsymbol{b}$, and **"det"** is the **determinant** of the matrix.

Example 18
Use Cramer's rule and solve the following system of equations:

$$\mathbf{2x + y = 9} \quad \text{and } \mathbf{3x - y = 11}$$

Solution
First write the system of equations in matrix form, then it looks as follows:

$$\begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = \begin{pmatrix} \mathbf{9} \\ \mathbf{11} \end{pmatrix}$$

Now it is in the form: $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where $\mathbf{A} = \begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{-1} \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} \mathbf{9} \\ \mathbf{11} \end{pmatrix}$

According to Cramer's rule:

$$x = \frac{\det \mathrm{A}_x}{\det \mathrm{A}} \quad \text{and} \quad y = \frac{\det \mathrm{A}_y}{\det \mathrm{A}}$$

$$\det \mathrm{A} = \begin{vmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{-1} \end{vmatrix} = (2)(-1) - (1)(3) = -5$$

$$\det \mathrm{A}_x = \begin{vmatrix} \mathbf{9} & \mathbf{1} \\ \mathbf{11} & \mathbf{-1} \end{vmatrix} = (9)(-1) - (1)(11) = -20$$

$$\det \mathrm{A}_y = \begin{vmatrix} \mathbf{2} & \mathbf{9} \\ \mathbf{3} & \mathbf{11} \end{vmatrix} = (2)(11) - (9)(3) = -5$$

Substitute back to calculate the values of x and y:

$$\therefore x = \frac{\det \mathrm{A}_x}{\det \mathrm{A}} = \frac{-20}{-5} = 4 \quad \text{and} \quad y = \frac{\det \mathrm{A}_y}{\det \mathrm{A}} = \frac{-5}{-5} = 1$$

Example 19

If $\det \mathbf{A} = 0$, we divide by 0, which is undefined. If it happens with Cramer's rule, it means that there are either no solution, or infinitely many solutions is.

Use Cramer's rule and calculate the value of $a > 0$ if the following system of equations has no solution:

$$2x - 2y - 2z = 4$$

$$ax + 2y + 2z = 7$$

$$-x + y + \mathrm{a}z = 1$$

Solution

It means that $\det \mathbf{A} = 0$

$$|A| = \begin{vmatrix} 2 & -2 & -2 \\ a & 2 & 2 \\ -1 & 1 & a \end{vmatrix} = 0$$

Therefore $2(2a - 2) + 2(a^2 + 2) - 2(a + 2) = 0$

$$2a - 2 + a^2 + 2 - a - 2 = 0$$

$a^2 + a - 2 = 0$, hence $a = 1$.
(Remember question: $\boldsymbol{a} > \mathbf{0}$)

COMPLEX NUMBERS

Complex numbers can be written in three ways. You must be able to convert from any of the three to the other. You should also be able to do calculations with these:

1. Rectangular form: addition, subtraction, multiplication, division.
2. Polar form: multiplication, division, exponents.
3. Exponential form: multiplication, division, exponents.

Polar form and exponential form do not need any calculations, it is just a different form of writing:

$$r(cos\theta + isin\theta) = re^{i\theta}$$

The left hand side is in polar form and the righthand side is in exponential form. Polar form can also be written in short as $rcis\theta$.

Convert from rectangular form to polar form:

Example 20

Convert $-\sqrt{3} + i$ to polar form (or exponential).

Solution

First you have to determine in which quadrant the number lies. Here x is negative and y is positive. Therefore the second quadrant.

Hence $\theta = \pi$ $-$angle(180° $-$hoek).

Angle= $arctan\left(\frac{y}{x}\right) = arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$.

Therefore $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

The value of r is determined by using Pythagoras:

$r^2 = x^2 + y^2$ and remember the sign does not matter:

$r^2 = \left(\sqrt{3}\right)^2 + 1^2 = 4$, therefore $r = 2$.

Polar form: $2cis(\frac{5\pi}{6})$. Exponential form: $2e^{i\frac{5\pi}{6}}$.

Convert polar form to rectangular form:

Example 21

Convert $\sqrt{2}cis(\frac{7\pi}{4})$ to rectangular form.

Solution

It is very simple, as long as you remember that x is associated with cos and y with sin:

$x = \sqrt{2}\cos\frac{7\pi}{4} = 1$ and $y = \sqrt{2}\sin\frac{7\pi}{4} = -1$ (do on calculator, make sure it is set to radians)

$$\therefore \sqrt{2}cis\left(\frac{7\pi}{4}\right) = 1 - i.$$

Calculations

Example 22

Determine $\frac{\mathbf{3-4}\boldsymbol{i}}{\mathbf{5-2}\boldsymbol{i}}$ such that the denominator is rational.

Solution

Here we multiply with the conjugate of the denominator over itself:

$$\frac{\mathbf{3-4}\boldsymbol{i}}{\mathbf{5-2}\boldsymbol{i}} \times \frac{\mathbf{5+2}\boldsymbol{i}}{\mathbf{5+2}\boldsymbol{i}} = \frac{(3-4i)(5+2i)}{(5-2i)(5+2i)} = \frac{23-14i}{29} = \frac{23}{29} - \frac{14}{29}i$$

With polar form the following rules hold:

- Do with r what the calculation asks.
- The angles are added/ subtracted if there is multiplication/ division.
- The angles are multiplied if there are exponents.

Example 23

Calculate the value of $\left(1-\sqrt{3}i\right)^{6}$. Do the calculation in exponential form and give the answer in rectangular form.

Solution

The quadrant is 4. It is then easier to work with $-\theta$.

$\theta = -arctan\left(\frac{\sqrt{3}}{1}\right) = -\frac{\pi}{3}$ and $r = 2$.

$\therefore \left(2e^{\frac{-i\pi}{3}}\right)^{6} = 2^{6}e^{\frac{-i\pi}{3}\times 6} = 64e^{-2i\pi}$

$x = 64\cos(-2\pi) = 64$ and $y = 64\sin(-2\pi) = 0$.

Answer therefore is 64.

Example 24

Calculate the value of $\frac{\left(1-\sqrt{3}i\right)^{2}}{-\sqrt{3}+i}$. Do the calculation in polar form and give the answer in rectangular form.

Solution

These two complex numbers were already converted to polar form in the previous example:

$$\frac{\left(2cis\left(-\frac{\pi}{3}\right)\right)^{2}}{2cis(\frac{5\pi}{6})} = \frac{4cis(-\frac{2\pi}{3})}{2cis(\frac{5\pi}{6})} = 2cis(-\frac{2\pi}{3} - \frac{5\pi}{6})$$

$= 2cis(-\frac{3\pi}{2})$

$\therefore x = 2\cos(-\frac{3\pi}{2}) = 0$ and $y = 2\sin(-\frac{3\pi}{2}) = 2$

Answer is $2i$

Roots of complex numbers

Example 24(b)
Solve: $x^3 = 1$.

Solution
We want to know wat $\sqrt[3]{1}$ is. According to Gauss there must be three answers. In algebra we always factorise if possible:
$x^3 - 1 = 0 \Rightarrow (x-1)(x^2+x+1) = 0 \Rightarrow x = 1$ or $x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ (use formula)

Now we are going to do the same problem using de Moivre's theorem. For this the number must be in polar form. First write it as a complex number :$z = 1 + 0i$ and then in polar form: $z = 1cis(0)$. We will get the cube root of z (or to the power of $\frac{1}{3}$)

It is geometrically true that you can add 360˚ or 2π radians to the angle and it stays the same number, because it will stay at the same place on a diagram. Let us do it two times, so that there are three numbers:

$z = 1cis(0) = z = 1cis(2\pi) = z = 1cis(4\pi)$. (Add 2π two times to previous angle)

These three numbers are exactly the same. Now de Moivre, get cube root of number and divide angle by 3:
$z = 1\,\mathrm{cis}\left(\frac{0}{3}\right) = 1\mathrm{cis}\left(\frac{2\pi}{3}\right) = z = 1cis\left(\frac{4\pi}{3}\right)$.

Root form:
$z = 1$ of $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ which are exactly the same.

TRIGONOMETRY

INVERSE FUNCTIONS

In Alpha mathematics all angles are in radians. With inverse functions x and y are swopped. In any trigonometric function x is an angle and y a number. With inverse functions these are swopped.

A problem with trigonometric functions is that they are not one to one functions. For any y value there are infinitely many x values. To make the inverse a function, the domain of the original trigonometric function has to be restricted, so that the inverse can be a function. This could have been done in many ways, but mathematicians have decided on a fixed system namely:

- $f(x) = sinx$: $x \in [-\frac{\pi}{2}; \frac{\pi}{2}]$ with $y \in [-1; 1]$
- $f(x) = cosx$: $x \in [0; \pi]$ with $y \in [-1; 1]$
- $f(x) = tanx$: $x \in [-\frac{\pi}{2}; \frac{\pi}{2}]$ with $y \in \mathbb{R}$

These x values become the y-values of the inverse functions. Furthermore, the y-values become the x values of the inverse functions. With this type of questions it helps to write down the domain and range of the original function, and then swop it for the inverse function.

All translations done with trigonometric functions in school mathematics, can also be done with inverse functions.

Example 25

Determine the inverse function of $f(x) = 2\sin(\mathrm{x} - \frac{\pi}{3})$

Solution

Swop x and y: $x = 2sin\ (y - \frac{\pi}{3})$, thus $sin(y - \frac{\pi}{3}) = \frac{x}{2}$

Then $y - \frac{\pi}{3} = arcsin\left(\frac{x}{2}\right), \therefore f^{-1}(x) = arcsin\left(\frac{x}{2}\right) + \frac{\pi}{3}$

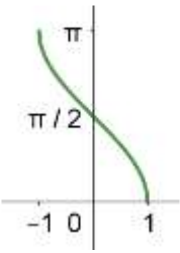
Example 26

Given: $h(x) = arccos\left(x - \frac{1}{2}\right) - \frac{\pi}{6}$

Sketch the graph of h. Clearly show all intercepts with the axes and coordinates of the endpoints of the function.

Solution

It is a $arccos$ graph that shifts ½ units to the right and $\frac{\pi}{6}$ units downward. The function $y = arccosx$ looks like this:



It intercepts the x-aaxis at (1; 0) and the y-axis at $(0;\frac{\pi}{2})$. The endpoints are at $(-1;\pi)$ and (1; 0).

All of these points must shift ½ units to the right and $\frac{\pi}{6}$ units downwards. The shape remains unchanged.

THUS: end points: C$(1\frac{1}{2};-\frac{\pi}{6})$, A $(-\frac{1}{2};\frac{5\pi}{6})$

The intercepts must be calculated:

y-intercept: $y = arccos\left(-\frac{1}{2}\right) - \frac{\pi}{6} = \frac{\pi}{2}$ (use calculator).

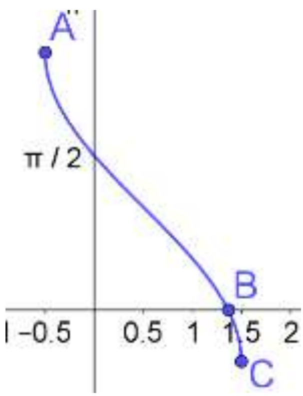
x-intercept:

$0 = arccos\left(x - \frac{1}{2}\right) - \frac{\pi}{6}$

Thus $arccos\left(x - \frac{1}{2}\right) = \frac{\pi}{6}$;

$\therefore x - \frac{1}{2} = \cos\left(\frac{\pi}{6}\right)$;

$x = \frac{1}{2} + \frac{\sqrt{3}}{2} \approx 1{,}4$ (B). Sketch:



APPLICATIONS OF RADIANS IN SECTORS

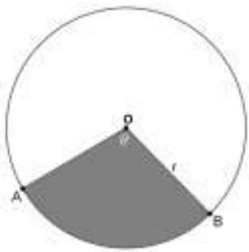
There are two formulas on the formula sheet, that are used:

$\boldsymbol{s = r\theta}$, where s is the arc length formed by radius r and angle θ.

$\boldsymbol{A = \frac{1}{2}r^2\theta}$, where A is the area of the sector formed by radius r and angle θ .

Example 27

The sketch shows a circle with centre O and radius r. A and B are points on the circumference of the circle and $A\hat{O}B = \theta$. The shaded sector has area of 96 and the arc length AB is 24.



(a) Calculate the radius of the circle.

(b) Calculate the size of angle θ

Solution

$\boldsymbol{S = r\theta} = 24$, $\theta = \frac{24}{r}$. Furthermore $\boldsymbol{A = \frac{1}{2}r^2\theta}$:

$A = \frac{1}{2}r^2\left(\frac{24}{r}\right) = 96$, $12r = 96$, $r = 8$.

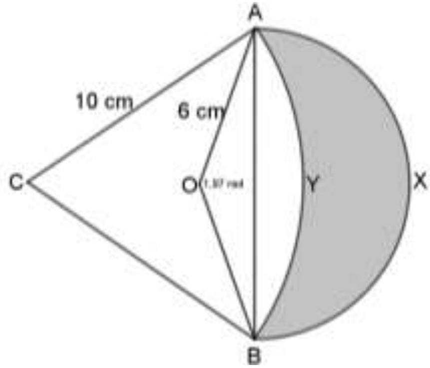
$\theta = \frac{24}{8} = 3$ rad.

Sometimes we have to use trigonometry rules for triangles.

Example 28

In the diagram OAXB is a sector of a circle with centre O and radius 6 cm. CAYB is a sector of another circle with centre C and radius of 10 cm.

$A\hat{C}B = \frac{\pi}{3}$ radians and $A\hat{O}B = 1{,}97$ radians.



(a) Show that the area of OAYB is 25,64 cm^2.

(b) Hence determine the area of AXBY, the shaded part.

Solution

(a) Sector ACB $= \frac{1}{2}(10^2)\left(\frac{\pi}{3}\right) = \frac{50\pi}{3} = 52{,}3598$

Use the sine rule:

$\Delta ACB = \frac{1}{2}(10^2)\sin\frac{\pi}{3} = 25\sqrt{3} = 43{,}3012$

$\Delta OAB = \frac{1}{2}(6^2)\sin(1{,}97) = 16{,}5846$

$\therefore$ Area $= 52{,}36 - 43{,}30 + 16{,}58 = 25{,}64$

(b) Sector OAXB $= \frac{1}{2}(6^2)(1{,}97) = 35{,}46$

Area of shaded part $= 35{,}46 - 25{,}64 = 9{,}82$

VECTORS

The following formulas are provided on the formula sheet:

$\lvert \boldsymbol{AB} \rvert = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$	$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
$\lvert \boldsymbol{OP} \rvert = \sqrt{a^2 + b^2 + c^2}$	$\boldsymbol{p} \bullet \boldsymbol{q} = \lvert \boldsymbol{p} \rvert \lvert \boldsymbol{q} \rvert \cos\theta$ $\boldsymbol{p} \bullet \boldsymbol{q} = x_1.x_2 + y_1.y_2 + z_1.z_2$
$\boldsymbol{a} \times \boldsymbol{b} = \lvert \boldsymbol{a} \rvert \lvert \boldsymbol{b} \rvert \sin\theta\,.\boldsymbol{n}$	$\alpha = \arccos\left(\frac{u_n}{\lvert \boldsymbol{u} \rvert}\right)$

Example 29

Determine the distance from point P(2; –1; 7) to Q(1;–3; 5).

Solution

$$\lvert \boldsymbol{PQ} \rvert = \sqrt{(1-2)^2 + (-3+1)^2 + (5-7)^2} = 3$$

Example 30

Determine the vector with starting point A (2; -3; 4) and endpoint B (-2; 1; 1). Also determine the magnitude of the vector.

Solution

$\boldsymbol{AB} = (-2-2\,;\ 1-(-3);\ 1-4) = (-4;\ 4;\ -3).$

Magnitude: $\lvert \boldsymbol{AB} \rvert = \sqrt{16+16+9} = \sqrt{41}$

Example 31

Determine the unit vector of the vector $\boldsymbol{u} = 2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$

Solution

The magnitude of the vector is $\sqrt{4+1+4} = 3$. Thus the unit vector, that is a vector in the same direction as $\boldsymbol{u}$ but with magnitude of 1, is equal to

$$\frac{1}{3}(2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}) = \left(\frac{2}{3}; -\frac{1}{3}; -\frac{2}{3}\right)$$

Example 32

Determine the angle between the vector and the x-aaxis.

Solution

$\alpha = arccos\left(\frac{2}{3}\right) = 0{,}84$ rad with the x-axis.

Example 33

Determine the dot product of vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, given that the angle between the two vectors is $\frac{\pi}{3}$ and

$|\boldsymbol{p}| = 25$ units and $|\boldsymbol{q}| = 4$ units.

Solution

$$\boldsymbol{p}.\boldsymbol{q} = |\boldsymbol{p}||\boldsymbol{q}|cos\theta = 25 \times 4 \times cos\ \frac{\pi}{3} = 50$$

Example 34

Determine the dot product of the vectors

$\boldsymbol{a} = 2\boldsymbol{i} + 5\boldsymbol{j} - 4\boldsymbol{k}$ and $\boldsymbol{b} = -2\boldsymbol{i} - 3\boldsymbol{j} - 5\boldsymbol{k}$

Note: The question could also have been:

$\boldsymbol{a} = (2;\ 5;\ -4)$ and $\boldsymbol{b} = (-2;\ -3;\ -5)$

Solution

$$\boldsymbol{a}.\boldsymbol{b} = (2 \times -2) + (5 \times -3) + (-4) \times (-5) = 1$$

Example 35

Determine the angle formed between the vectors

$\boldsymbol{p} = 4\boldsymbol{i} + 0\boldsymbol{j} + 7\boldsymbol{k}$ and $\boldsymbol{q} = -2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$

Solution

We have to calculate $\boldsymbol{p}.\boldsymbol{q}$ as well as $|\boldsymbol{p}|$ and $|\boldsymbol{q}|$.

$\boldsymbol{p}.\boldsymbol{q} = 4.(-2) + 0.1 + 7.3 = 13$

$|\boldsymbol{p}| = \sqrt{4^2 + 0^2 + 7^2} = \sqrt{65}$ and

$|\boldsymbol{q}| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$.

Therefore $\theta = arccos\frac{13}{\sqrt{65}.\sqrt{14}} = 1{,}13$ radians.

Example 36

Determine $\boldsymbol{a} \times \boldsymbol{b}$ if $\boldsymbol{a} = (2;\ 3;\ 4)$ and $\boldsymbol{b} = (5;\ 6;\ 7)$

Solution

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{vmatrix} = \boldsymbol{i}(21-24) - \boldsymbol{j}(14-20) + \boldsymbol{k}(12-15)$$

$$= -3\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$$

Example 37

Given $\boldsymbol{a} = (-3;\ 1;\ -7)$ and $\boldsymbol{b} = (0; -5; -5)$. First determine if the vectors are parallel, and if not, the area of the parallelogram formed by these vectors.

Solution

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= \boldsymbol{i}(-5-35) - \boldsymbol{j}(15-0) + \boldsymbol{k}(15-0)$$

$$= -40\boldsymbol{i} - 15\boldsymbol{j} + 15\boldsymbol{k}$$

Since it is not **0**, the vectors are not parallel, and it can form a parallelogram.

To determine the area of the parallelogram between these two vectors, we calculate the magnitude of the cross product:

Area parm $= \sqrt{1600 + 225 + 225} = 5\sqrt{82}$

Example 38

Given the vectors $\boldsymbol{a} = (2; 3; -1)$ and $\boldsymbol{b} = (0; -2; 5)$. Determine a vector perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$.

Solution

The vector perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\boldsymbol{a} \times \boldsymbol{b}$

And $\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 3 & -1 \\ 0 & -2 & 5 \end{vmatrix}$

$$= \boldsymbol{i}(15-2) - \boldsymbol{j}(10-0) + k(-4-0)$$

$$= 13\boldsymbol{i} - 10\boldsymbol{j} - 4\boldsymbol{k}$$

DIFFERENTIATION

LIMITS

The following limits are used to determine the horizontal asymptotes:

$\lim_{x\to\infty}\frac{a}{x}=0$ and $\lim_{x\to\infty}\frac{2x^2-1}{x-x^2}=\lim_{x\to\infty}\frac{x^2(2-\frac{1}{x^2})}{x^2(\frac{1}{x}-1)}=-2.$

For continuity limits must be determined from left and right. This will be explained at continuity.

Further if $\lim_{x\to 0}\frac{a}{x}=\pm\infty$, it does not exist.

CONTINUITY

The notation in this section is very important. Practise so that you write it correctly. Most marks are lost as a result of incorrect notation.

A function is continuous at a point a when

$$\boldsymbol{\lim_{x\to a} f(x)=f(a)}$$

This means that three things should be true:

1. The limit must exist. This implies that the limit from the left must be equal to the limit from the right. If this is not true, there is a jump discontinuity at a .
   If the limit is equal to infinity, there is a asymtotic discontinuity, as with the hyperbola.
2. There must be a x-value in order for the y-value to exist. If this is not true, there is a removable discontinuity.
3. Number 1 must be equal to number 2. If this is not true, there is a removable discontinuity.

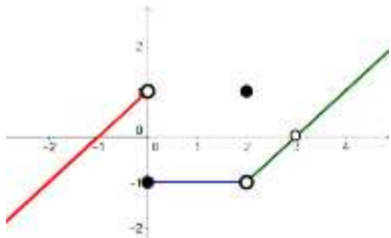
All three the conditions should be true for a function to be continuous at a specific point. Only **one** has to be true for a function to be discontinuous.

To **TEST for continuity,** the following three tests has to be done:

1. Does $\lim_{x \to a} f(x)$ exist, in other words is $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$?
(The – is left and the +right)
2. Does $f(a)$ exist, in other words, is there a y- and x-value defined?
3. Is $\lim_{x \to a} f(x) = f(a)$? (*this means:* Is the answer of nr. 1 equal to the answer of nr. 2?)

These problems can be asked to be done by calculations or with a sketch.

Example 39
Consider the sketch of a function:



Name the points where the function is discontinuous, name the type and give an algebraic reason for the discontinuity.

Solution
$x = 0$, jump discontinuity,
$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$.

$x = 2$, removable, $\lim_{x \to 2} f(x) \neq f(2)$.

$x = 3$, removable, $f(3)$ does not exist.

Pay attention to the notation of the algebraic reasons.

Example 40

Consider $f(x) = \begin{cases} x+4 & \text{as} \quad x<0 \\ 4 & \text{as} \quad x=0 \\ x^2+4 & \text{as} \quad x>0 \end{cases}$

Determine if $f(x)$ is continuous at $\boldsymbol{x=0}$.

Solution

Point 1: $\lim_{x \to 0^-} f(x) = 4$ and $\lim_{x \to 0^+} f(x) = 4$

This means that: $\lim_{x \to 0} f(x) = \mathbf{4}$

Point 2: The function-value at the point $\boldsymbol{x=4}$

$\therefore f(\mathbf{0}) = \mathbf{4}$.

Point 3: $\therefore \lim_{x \to 0} f(x) = f(0)$

The function is therefore **continuous** at the point $\boldsymbol{x=0}$**.**

Example 41

$$f(x) = \begin{cases} \boldsymbol{\cos x} & \text{if} \quad x < \frac{\pi}{2} \\ \boldsymbol{x^2 + c} & \text{if} \quad x \geq \frac{\pi}{2} \end{cases}$$

Determine the value of c so that f will always be **continuous.**

Solution

If f is **continuous**, the limit must exist at $x = \frac{\pi}{2}$.

This means that the graphs of $\boldsymbol{\cos x}$ and $\boldsymbol{x^2 + c}$ must meet at $x = \frac{\pi}{2}$. To "meet", the two functions must have the same value at $x = \frac{\pi}{2}$:

$$\lim_{x \to \left(\frac{\pi}{2}\right)^-} \boldsymbol{f(x)} = \lim_{x \to \left(\frac{\pi}{2}\right)^+} \boldsymbol{f(x)}$$

$$\therefore \cos\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^2 + c$$

$$\therefore 0 = \left(\frac{\pi}{2}\right)^2 + c \quad \Longrightarrow \quad c = -\frac{\pi^2}{4}$$

DIFFERENTIABILITY

For a function to be differentiable at a point a the gradient from the left must be equal to the gradient from the right. It should also be **continuous** at the point. Algebraically it is written as:

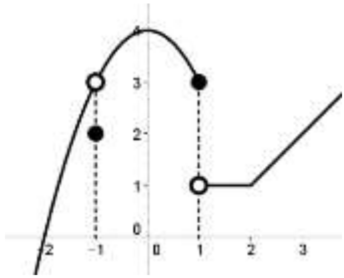
$$\lim_{x \to a^-} f'(x) = \lim_{x \to a^+} f'(x)$$

If the function is continuous we just need to test for differentiability at both sides. Just check that the function is continuous at the point.

Example 42

Consider the sketch and answer the questions:

Give all the points where:



1 The left and right limit are not equal.

2 The limit and the function value exist, but are not equal.

3 The function is continuous but not differentiable.

4 The function is not differentiable.

Solution

The question is asking only for the points and not for algebraic reasons.

1 $x = 1$

2 $x = -1$

3 $x = 2$

4 All the points mentioned above.

Example 43

Given: $f(x) = \begin{cases} x-5 & \text{if} \quad x<2 \\ 3 & \text{if} \quad 2 \leq x \leq 4 \\ \sqrt{x+5} & \text{if} \quad x>4 \end{cases}$

Answer the following questions:

1 Is $f(x)$ continuous at $x = 1$? Motivate.

2 Is $f(x)$ continuous at $x = 4$? Motivate.

3 Is $f(x)$ differentiable at $x = 1$ and $x = 3$? Motivate.

Solution

1 No, $\lim_{x\to\mathbf{2}^-} f(x) = -3$ and $\lim_{x\to\mathbf{2}^+} f(x) = 3$.

2 Yes, $\lim_{x\to\mathbf{4}^-} f(x) = 3$ and $\lim_{x\to\mathbf{4}^+} f(x) = 3$, $f(\mathbf{4}) = 3$, therefore $\lim_{x\to\mathbf{4}} f(x) = f(\mathbf{4})$.

3 $x = 1$: No, it is not continuous.

$x = 4$: $\lim_{x\to\mathbf{4}^-} f'(x) = 0$ en $\lim_{x\to\mathbf{4}^+} f'(x) = \frac{1}{2}(4+5)^{-\frac{1}{2}}$

No, not differentiable.

Example 44

Given: $f(x) = \begin{cases} \mathbf{2x + a} & \text{if} \quad x \le 3 \\ \mathbf{bx^2 + 4} & \text{if} \quad x > 3 \end{cases}$

Determine the values of a and b so that the function is differentiable for all values x.

SOLUTION

For a funtion to be differentiable it must also be continuous.

Hence f must be **continuous** at the point $\mathbf{x = 3}$:

$$\therefore \lim_{x \to \mathbf{3}^-} \mathbf{f(x)} = \lim_{x \to \mathbf{3}^+} \mathbf{f(x)}$$

$$2(3) + a = b(3)^2 + 4$$

$$a - 9b = -2$$

and f must be **differentiable** at the point $\mathbf{x = 3}$:

$$f'(x) = \begin{cases} \mathbf{2} & \text{if} \quad x \le 3 \\ \mathbf{2bx} & \text{if} \quad x > 3 \end{cases}$$

Derive to get $f'(x)$

$$\lim_{x \to \mathbf{3}^-} \mathbf{f'(x)} = \lim_{x \to \mathbf{3}^+} \mathbf{f'(x)}$$

$$2 = 2b(3)$$

$$b = \frac{1}{3}$$

Therefore $a - 9\left(\frac{1}{3}\right) = -2 \quad \Longrightarrow \quad a = 1$

USE OF DIFFERENTIATION RULES

These rules are given in the table for derivatives of functions.

Product rule: $\frac{\boldsymbol{d}}{\boldsymbol{dx}} f(x).g(x) = f'(x).g(x) + f(x).g'(x)$

Quotient rule: $\frac{\boldsymbol{d}}{\boldsymbol{dx}} \frac{f(x)}{g(x)} = \frac{f'(x).g(x) - f(x).g'(x)}{[g(x)]^2}$

Chain rule: $\frac{\boldsymbol{d}}{\boldsymbol{dx}} f[g(x)] = f'[g(x)].g'(x)$

Example

Determine $\frac{d}{dx}[\sin x.(5x^2 + 2x)]$

Solution

$f(x) = \sin x \qquad g(x) = 5x^2 + 2x$

$f'(x) = \cos x \qquad g'(x) = 10x + 2$

$$\therefore \frac{d}{dx}[\sin x.(5x^2 + 2x)]$$

$$= \cos x\,(5x^2 + 2x) + \sin x\,(10x + 2)$$

All steps are explained here, but in the exam only the answer can be given. This is also applicable for the next example.

Example 45

Determine: $D_x\left[\frac{\sin x}{2x+1}\right]$

Solution

$f(x) = \sin x \qquad f'(x) = \cos x$

$g(x) = 2x + 1 \qquad g'(x) = 2$

$$\therefore D_x\left[\frac{\sin x}{2x+1}\right] = \frac{\cos x.(2x+1) - \sin x.2}{(2x+1)^2}$$

Note the notation. Marks are often lost for incorrect notation.

Example 46

Determine $\frac{d}{dx}[\tan(\boldsymbol{2x^3 - x})]$

Solution

$\sec^2(\boldsymbol{2x^3 - x}) \times (\boldsymbol{6x^2 - 1})$

EXPONENTS AND LOGARITHMS

The following formulae is on the formula sheet:

$\frac{d}{dx}(a^x) = a^x . lna$ and $\frac{d}{dx}(\log_a x) = \frac{1}{x.lna}$

Two special cases are e^x and lnx. It can be deducted from the formulae, but remember $lne = 1$:

$\frac{d}{dx}(e^x) = e^x . lne = e^x$ and

$$\frac{d}{dx}(lnx) = \frac{d}{dx}\log_e x = \frac{1}{x.lne} = \frac{1}{x}$$

When expressions with logarithms must be derived, it is **ALWAYS** easier to expand the expression using the laws of logarithms.

Example 47

Determine $f'(x)$ if $f(x) = 2^{5x} + \log\frac{x^2+1}{\sin^5 x}$

Solution

Expand the logarithm part:

$$f(x) = 2^{5x} + \log(x^2 + 1) - 5\log(\sin x)$$

$$f'(x) = 2^{5x}.5.ln2 + \frac{1}{(x^2+1).ln10}.2x - \frac{5}{\sin x . ln10}.\cos x$$

IMPLICIT DIFFERENTIATION

This is used where you cannot isolate y. You may use the notation y' in the place of $\frac{dy}{dx}$ here. Be aware of the product and quotient rules and remember the following:

function	constants	x	y	xy	y^2	$\sin y$	π
derivative	0	1	y'	$y + xy'$	$2y.y'$	$\cos y.y'$	0

The process is as follows:

- Derive each term.
- Collect the terms with y' on one side.
- Take y' out as a common factor.
- Divide both sides with the factor in brackets.

This part is often combined with tangents. Remember $\boldsymbol{m = y'}$.

Then use the formula $\boldsymbol{y - y_1 = m(x - x_1)}$.

Example 48

Use implicit differentiation and determine $\frac{dy}{dx}$ and the equation of the tangent to the curve $y^2e^{2x} = 3y + x^2$ at the point (0; 3).

Solution

Start by deriving, be careful for the first term which is a product:

$$2y.y'.e^{2x} + y^2.e^{2x}.2 = 3y' + 2x$$

All terms with y' collected:

$$2y.y'.e^{2x} - 3y' = 2x - y^2.e^{2x}.2$$

Take y' out as a common factor:

$$y'(2y.e^{2x} - 3) = 2x - y^2.e^{2x}.2$$

Divide: $y' = \frac{2x - y^2.e^{2x}.2}{2y.e^{2x} - 3}$

Deriving is done and now the equation of the tangent should be determined. Substitute (0; 3) in y' for m:

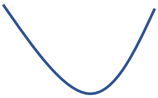
$$m = y' = \frac{0 - 9.e^0.2}{6.e^0 - 3} = -\frac{18}{3} = -6$$

$$y - y_1 = m(x - x_1),$$
$$y - 3 = -6(x - 0), \therefore y = -6x + 3$$

HIGHER ORDER DERIVATIVES

When a function is derived the first derivative exists, $f'(x)$. When the first derivative is derived, the second derivative exists, $f"(x)$.

$f'(x)$ gives the value of the gradient at any point on the graph of $y = f(x)$. If $f'(x)$ is positive, the graph is increasing. $f"(x)$ gives the value of the gradient at any point on the graph of $y = f'(x)$. The second derivative provides information on the shape of $f(x)$. If $f"(x)$ is positive, the graph of the gradient is increasing. This is called concave up.



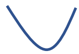
Note how the gradient increases.

If $f"(x)$ is negative, the graph's gradient becomes smaller. This is called concave up:

Graphs can have stationary points and points of inflection. You have to know the difference between the two, on the graph and algebraicly.

A **stationary point** develops when $f'(x) = 0$. It can have one of 3 forms:

Minimum turning point, where $f"(x) > 0$



Maximum turning point, where $f"(x) < 0$

Point of inflection, where $f"(x) = 0$

A point of inflection develops when $f"(x) = 0$, but $f"$ must change from positive to negative or negative to positive over this point. If this does not happen, there is no point of inflection.

There are two types of points of inflection:

A point which is also stationary, where $f'(x) = 0$.

The points of inflection which are not stationary points, where $f'(x) \neq 0$. This type can be clearly seen in the cubic graphs.

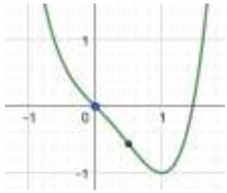
This is obvious not a stationary point.

These concepts have to be studied thoroughly.

Example 49
Given $f(x) = x^4 - x^3 - x$. Determine all possible stationary and turning points of f. Also state what type of stationary, if any, point it is.

Solution
Stationary: $f'(x) = 0$, therefore $4x^3 - 3x^2 - 1 = 0$
$x = 1$ is a solution, hence $(x-1)(4x^2 + x + 1) = 0$
$x = 1$ is the only real solution.
To determine the type, the second derivative must be determined and check if it is positive or negative at $x = 1$:
$f''(x) = 12x^2 - 6x$; $f''(1) = 12 - 6 = 6 > 0$
There is therefore a minimum turning point at $x = 1$.
Point of inflection: $f''(x) = 0$, thus $12x^2 - 6x = 0$
$\therefore 6x(2x-1) = 0$; hence $x = 0$ or $x = \frac{1}{2}$.
Test the sign of f'' at both sides of these points:
$x = 0$: $f''(-1) = 18 > 0$ and $f''\left(\frac{1}{4}\right) = -\frac{3}{4} < 0$.
Different.
$x = \frac{1}{2}$: $f''\left(\frac{1}{4}\right) = -\frac{3}{4} < 0$ and $f''(1) = 6 > 0$. Different.

The function has a minimum stationary point (turning point) at $x = 1$ and points of inflection at $x = 0$ and $x = \frac{1}{2}$.
Take note that these 2 points of inflection are not stationary. The graph will look like this, points of inflection indicated:



NEWTON'S METHOD

Newton's method is a way to determine the zero points or x-intercepts of a function, if it cannot be determined with the normal methods. The point of intersection of two graphs can also be determine using this method when the two equations are written as one. The formula for Newton's method is on the formula sheet:

$$\boldsymbol{a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}}$$

In the exam you must determine the derivative and then substitute everything in the formula. It's not necessary to write down all the approximations, just the final answer. Make sure how many decimal digits are asked.

Example 50

Use Newton's method and determine the x-values, rounded off to 4 digits, of the point of intersection of the graphs

$y = \tan(2x)$ and $y = \sqrt{x}$ in the interval $x \in (\frac{3\pi}{4}; \frac{5\pi}{4})$.

Use $x = \pi$ as first approximation.

Solution

Create a function: $f(x) = \tan(2x) - \sqrt{x}$. These function`s zero point will be the intersection of the two graphs.

$$f'(x) = \sec^2(2x).2 - \frac{1}{2}x^{-\frac{1}{2}}$$

Thus $a_{n+1} = a_n - \frac{\tan(2a_n) - \sqrt{a_n}}{\sec^2(2a_n).2 - \frac{1}{2}a_n^{-\frac{1}{2}}}$ with $a_n = \pi$.

Key in π = on your calculator. Type in the formula and use the “Ans” key everytime for a_n. Press the = key until your answer stays the same and then write it down, rounded off to the correct decimal digits:

$$x \approx 6{,}8865$$

OPIMISING

This is the same as the optimising of normal school mathematics, where the maximum or minimum value of an expression is obtained. The only difference is that the function can be any function which is derivable in Alpha Mathematics. The principle is the same: Derive, equate to zero and solve. It is usually not necessary to indicate whether it is a maximum or minimum value – just when it is specifically asked for. Where the minimum/ maximum value is required, the answer must be substituted in the original function.

Example 51

Determine the minimum value of $f(x) = e^{2x} - x$

Solution

$f'(x) = 2e^{2x} - 1 = 0$. Thus $e^{2x} = \frac{1}{2}$; $\therefore 2x = ln(\frac{1}{2})$

Giving $x = \frac{ln\left(\frac{1}{2}\right)}{2} \approx -0{,}35$. Hence the minimum of 0,85.

SKETCH OF RATIONAL FUNCTIONS

A rational function is a function in fraction form with variables in the denominator. Examples include:

$y = \frac{2}{x} - 4$; $y = \frac{x^2-x}{x+1}$

These functions normally have asymptotes, sometimes turning points and intercepts with the axes. All three must be determined to sketch the function. Firstly draw the graphs of the asymptote(s). Then mark the possible turning points and intercepts. If this is not enough to draw the function's graph, use the "Mode-Table" of your calculator and find a few more points on the graph. Remember the following about asymptotes:

- The vertical asymptote is when the function is undefined, there where division by zero would take place. This asymptote can NEVER intersect the function's graph. The vertical asymptote can be calculated with the formula $\lim_{x \to a} f(x) = \pm\infty$. You don't have to know this formula, but should understand what it stands for.
- The function can have either a horizontal or an oblique asymtote but NEVER both. These asymptotes can be intersected by the function, near $x = 0$, as it actually forms if $x \to \pm\infty$.
- The function has an oblique asymptote when the degree of the numerator is more than the degree of the denominator. If this is equal or smaller, then there is a horizontal asymptote. The horizontal asymptote can be calculates with the formula $\lim_{x \to \pm\infty} f(x) = b$.

  The oblique asymptote can be calculated with the formula $\lim_{x \to \pm\infty} (f(x) - (mx + c) = 0$. You also do not have to know these formulae, but should understand the meaning.

Example 52

Sketch $f(x) = \frac{x^2-x}{x+1}$.

Solution

Asymptotes:

Vertical where $x + 1 = 0$, so $x = -1$.

The degree of the numerator is bigger than that of the denominator, hence oblique. Use long or synthetic division: $(x^2 - x) \div (x + 1) = x - 2 + \frac{2}{x+1}$. The oblique asymptote is $y = x - 2$.

Turning points:

$$f'(x) = 0, \qquad \therefore \frac{(2x-1)(x+1) - (x^2 - x).1}{(x+1)^2} = 0$$

Giving $x^2 + 2x - 1 = 0$. Using the formula or calculator: $x = -2{,}41$ or $x = 0{,}41$. Substitute in f :
$y = -5{,}83$ or $y = -0{,}17$.

Hence (-2,41; -5,83) and (0,41; -0,17)

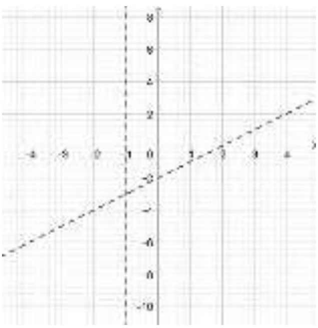
Intercepts:

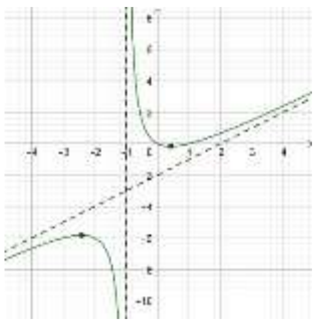
For x: $\frac{x^2-x}{x+1} = 0$, hence $x(x-1) = 0$; $\therefore x = 0$ or $x = 1$

For y: $y = \frac{0-0}{0+1} = 0$

Sketch:

Start with asymptotes and then add the rest:





Example 53

The following is known about the function $y = f(x)$:

$\lim\limits_{x \to 2} f(x) = \pm\infty$ and $\lim\limits_{x \to -2} f(x) = \pm\infty$, $\lim\limits_{x \to \pm\infty} f(x) = 2$.

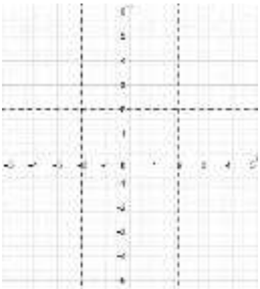
The function has a maximum turning point at (0; -1). It intersects the y-axis at $y = -1$ and has no x-intercepts.

Draw a sketch graph of $y = f(x)$.

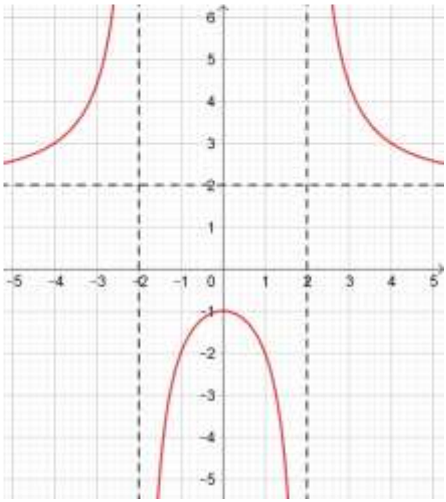
Solution

Start with the asymptotes:

Vertical at $x = \pm 2$ and horizontal at $y = 2$:



Plot the turning point and y-intercepts. No x-intercept. The graph may not intercept the vertical asymptote. The turning point is a maximum. Draw the graph:



INTEGRATION

TABLE

To use the table for integration, your should work from right to left. It also works if the x (in the table) is replaced with a linear function and you then divide the answer by the derivative of the linear function. However, the x may not be replaced by any other function, other methods have to be followed.

Example 54

Determine $\int \sin 5x \, dx$

Solution

$\mathbf{\sin x}$ is on the right on the formula sheet:

$\cos x$	$-\sin x$

However the $\boldsymbol{x}$ is now $\mathbf{5x}$, a linear function. The derivative is 5. The answer is:

$$\int \sin 5x \, dx = \frac{-\cos 5x}{5} + c$$

TRIGONOMETRIC IDENTITIES

When there are trigonometric functions in the integral, but not like in the previous example in the right hand side of the table, identities can be used. The following identities on the formula sheet are those used in integration:

$$\tan^2 x + 1 = \sec^2 x \qquad \cot^2 x + 1 = \operatorname{cosec}^2 x$$

$$\cos^2 x = \tfrac{1}{2}[1 + \cos(2x)] \qquad \sin^2 x = \tfrac{1}{2}[1 - \cos(2x)]$$

$$\sin A . \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\sin A . \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$$

$$\cos A . \cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]$$

Example 55

Determine the following integrals:

1 $\int \tan^2 x \, dx$ 2 $\int \cos^2 x \, dx$ 3 $\int \sin 5x . \cos 2x \, dx$

Solutions

1 Here the identity $\tan^2 x + 1 = \sec^2 x$ is used as there is no $\tan^2 x$ in the right table and there is a $\sec^2 x$:

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + c$$

2 The identity $\cos^2 x = \tfrac{1}{2}[1 + \cos(2x)]$ is used because the function becomes linear:

$$\int \cos^2 x \, dx = \int \frac{1}{2}[1 + \cos(2x)] \, dx = \frac{1}{2}[x + \frac{\sin 2x}{2}] + c$$

3 Use the identity

$\sin A . \cos B = \tfrac{1}{2}[\sin(A + B) + \sin(A - B)]$ to change to a linear function:

$$\int \sin 5x . \cos 2x \, dx = \int \frac{1}{2}[\sin(5x + 2x) + \sin(5x - 2x)] \, dx$$

$$= \tfrac{1}{2}[-\tfrac{\cos 7x}{7} - \tfrac{\cos 3x}{3}] + c$$

FACTOR INTEGRATION

This is also known as piece-wise integration. There isn't a product rule for integration as for differentiation. This method can be used where the product of two functions must be integrated. The following formula, on the formula sheet, is used:

$$\int f(x).g'(x)\,dx = f(x).g(x) - \int f'(x).g(x)\,dx + c$$

What is very important here is that one of the two functions must be integratable, as the g' on the left becomes the g on the right, which is the integral. This is important as you have to choose what is f and what is g'. A reasonable general rule is that if one of the functions is x, choose that to be f, because you have to derive the f and the derivative of x is 1.

Example 56

Determine $\int x.\sin x\,dx$

Solution

Decide on f en g'. The $\sin x$ can integrate, so:

$\boldsymbol{f(x) = x}$ en $\boldsymbol{g'(x) = \sin x}$.

Then $\boldsymbol{f'(x) = 1}$ en $\boldsymbol{g(x) = -\cos x}$

$$\int \boldsymbol{f(x)}.\boldsymbol{g'(x)}\,dx = \boldsymbol{f(x)}.\boldsymbol{g(x)} - \int \boldsymbol{f'(x)}.\boldsymbol{g(x)}\,dx + c$$

And $\int \boldsymbol{x}.\boldsymbol{\sin x}\,dx$ = $\boldsymbol{x}.(\boldsymbol{-\cos x}) - \int \boldsymbol{1}.\boldsymbol{-\cos x}\,dx$

= $-x.\cos x + \sin x + c$

INTEGRATION WITH PARTIAL FRACTIONS

With these questions it will always be stated that you have to use partial fractions.

Example 57

Determine $\int \frac{x^2+3}{(x-1)(x^2+1)} dx$ by using partial fractions.

Solution

Let $\frac{x^2+3}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$$x^2 + 3 = A(x^2 + 1) + (Bx + C)(x - 1)$$

Let $x = 1$: $4 = A(2); A = 2$

Use x^2: $1 = A + B$, thus $B = -1$

Use constants: $3 = A - C$, thus $C = -1$

so $\int \frac{2}{x-1} dx - \int \frac{x+1}{x^2+1} dx$ Separate the 2nd fraction:

$\int \frac{2}{x-1} dx - \int (\frac{x}{x^2+1} + \frac{1}{x^2+1}) dx$

$= 2ln(x - 1) - \frac{ln(x^2+1)}{2} - \arctan x + c$

The yellow highlighted part is discussed in the next section.

INTEGRATION WITH SUBSTITUTION

These integrals is in the form of a fraction. There is always two functions and the one is the derivative of the other. Substitution can be used, but it can be done without. It is explained here with the use of substitution.

Example 58

Determine $\int \frac{x}{x^2+1} dx$ (This is from example 57)

Solution

The x is the derivative of $x^2 + 1$, with constants which will be sorted out. Always let the function of which the derivative is there be equal to u. Now we will derive u with reference to x. This will give the second function. Substitute the integral – you will be correct if there appear no x in.

So $\boldsymbol{u = x^2 + 1}$. Derive, $\frac{du}{dx} = 2x$ and make dx the object:

$$du = 2x.dx, \therefore \boldsymbol{dx = \frac{du}{2x}}$$

Substitute in the integral:

$\int \frac{x}{\boldsymbol{x^2+1}} \boldsymbol{dx}$ becomes $\int \frac{x}{\boldsymbol{u}} \frac{\boldsymbol{du}}{\boldsymbol{2x}} = \int \frac{1}{2u} du$

This is correct because we have no x.

$$= \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} lnu + c$$

Substitute $u = x^2 + 1$: Answer $\frac{1}{2} ln(x^2 + 1) + c$

AREA WITH INTEGRATION

The area between a graph an the x-axis between two points can be calculated with a definite integral. When the area between two graphs must be determined, subtract the equation of the bottom one from the equation from the top one and determine the definite integral between the intersections of the graphs. If you cannot determine which graph is the top one, choose any of the two. Should your answer be negative then you have chosen the wrong one. You can just change the answer to the positive value. An area below the x-axis is always negative. Where the graph intersects the x-axis, it should be calculated as two separate definite integrals. You don't have to do this where you are calculating the area between two graphs. Often in these questions the area is given and you are aksed to calculate the value of one of the points.

Example 59

The area between the graphs $f(x) = (x + 2)^2$ and $g(x) = \frac{1}{x+2}$ between the points $x = -1$ and $x = 1$ is equal to $\frac{26}{a} - lna$. Determine the value of a. Show all your steps.

Solution

Area = $\int_{-1}^{1}[(x + 2)^2 - \frac{1}{x+2}]\, dx$

= $\frac{1}{3}(x + 2)^3 - ln(x + 2)\Big|_{-1}^{1}$

= $\frac{1}{3}(3)^3 - ln3 - (\frac{1}{3}(1)^3 - ln1)$

= $9 - ln3 - \frac{1}{3} = \frac{26}{3} - ln3$. Hence $a = 3$

VOLUME OF ROTATING BODIES

The formula used here is on the formula sheet:

$V = \pi \int_a^b [f(x)]^2\, dx$

When the volume between two graphs must be determined, each one should be calculated separate and the answers subtracted from one another.

Example 60

The graph of $f(x) = \sqrt{2x-4}$ rotates around the x-axis, between $x = p$ and $x = 10$. The volume of the rotation body created is equal to 64π. Determine the value of p.

Solution

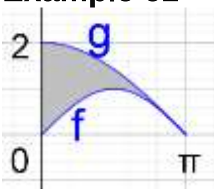
$V = \pi \int_a^b [f(x)]^2\, dx$, and $[f(x)]^2 = \sqrt{2x-4}^2 = 2x - 4$

So $V = \pi \int_p^{10} 2x - 4\, dx = 64\pi$

And $x^2 - 4x \Big|_p^{10} = 100 - 40 - (p^2 - 4p) = 64$

$p^2 - 4p + 4 = 0$, and $p = 2$

Example 61



The sketch shows the graphs of $f(x) = \sin x$ and $g(x) = 2\cos\frac{x}{2}$. Determine the volume of the solid formed when the shaded area between the graphs is rotating around the x-axis. Show all steps.

Solution

$V = \pi \int_0^{\pi} \left(\left[2\cos\frac{x}{2}\right]^2 - [\sin x]^2 \right) dx$

$= \pi \int_0^{\pi} \left(\frac{4.1}{2}[1 + \cos(2x)] - \frac{1}{2}[1 - \cos(2x)] \right) dx$

$= \frac{\pi}{2}\left[4\left(x + \frac{1}{2}\sin 2x\right) - \left(x - \frac{1}{2}\sin 2x\right)\right] \Big|_0^{\pi} = \frac{\pi}{2}(3\pi - 0)$

$= \frac{3\pi^2}{2}$

THE RIEMANN SUM

This is used to calculate the area between a graph and the x-axis, or the value if an integral, without integrating. What happens here us that the area is divided in a lot of rectangles with equal widths. The areas is calculated and then the number of rectangles is increased to an infinite number of rectangles. This will give the real area. The formula on the formula sheet is:

Riemannsom / *Riemann sum* $= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i).\Delta x_i$

The following formulae are also used:

$\sum_{i=1}^{n} 1 = n$ $\quad \sum_{i=1}^{n} i = \frac{n^2}{2} + \frac{n}{2}$ $\quad \sum_{i=1}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$

The problems can be divided into 6 steps:

STEP 1

Determine the width of each rectangle. There is n divisions in the interval $[a; b]$, so each width will be $(b - a) \div n$. All the rectangles will have exactly the same width. It is written as follows:

$$\Delta x_i = \frac{b - a}{n}$$

STEP 2

Determine the value of the i'th rectangle on the x-axis:

$$\boldsymbol{x_i = Starting\ point + width \times i}$$

$$\boldsymbol{x_i = a + \frac{b-a}{n} i = a + \Delta x_i . i}$$

STEP 3

Determine the height (length) of each rectangle.
This is the y-value at each partition point.
Because the points are x_i the $\boldsymbol{y}$-values is $\boldsymbol{f(x_i)}$.

STEP 4

You have to realise here that Δx_i does not have a i in and hence can be taken out commonly. Determine the sum of all heights.This is written as:

$$\sum_{i=1}^{n} f(x_i)$$

STEP 5

Multiply the width of all the rectangles with the sum of the heights:

Area = width x sum of the heights

= **STEP 1** x **STEP 4**

$$\Delta x_i \times \sum_{i=1}^{n} f(x_i)$$

Use the formulae from the formula sheet, it is given at the beginning, to simplify:

STEP 6

An infinite number of rectangles:

$$\int_a^b f(x)\,dx$$

$$= \lim_{n\to\infty}\left[width \times \sum heights\right]$$

$$= \lim_{n\to\infty} \textbf{STEP 5}$$

So the formula for the Riemann Sum:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i).\Delta x_i$$

Which we can change to

$$\lim_{n\to\infty} \frac{b-a}{n} \times \sum_{i=1}^{n} f(x_i)$$

In this step everything with a n in the denominator will become 0. There can be no n in the denominator. If there is one, you made a mistake.

Example 62

Use a Riemann sum and determine $\int_0^2 -x^2 + x + 2\,dx$

Solution

Step 1: $\Delta x_i = \frac{2-0}{n} = \frac{2}{n}$

Step 2: $x_i = 0 + \Delta x_i . i = \frac{2i}{n}$

Step 3: $f\left(\frac{2i}{n}\right) = -\left(\frac{2i}{n}\right)^2 + \frac{2i}{n} + 2 = -\frac{4i^2}{n^2} + \frac{2i}{n} + 2$

Step 4:

$$\sum_{i=1}^{n} f(x_i) = \sum_{i=1}^{n} -\frac{4i^2}{n^2} + \sum_{i=1}^{n} \frac{2i}{n} + \sum_{i=1}^{n} 2$$

Take out everything without i out as a common:

$$\sum_{i=1}^{n} f(x_i) = -\frac{4}{n^2}\sum_{i=1}^{n} i^2 + \frac{2}{n}\sum_{i=1}^{n} i + 2\sum_{i=1}^{n} 1$$

Use the formulae:

$= -\frac{4}{n^2}(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}) + \frac{2}{n}\left(\frac{n^2}{2} + \frac{n}{2}\right) + 2(n)$

Step 5:

$$\Delta x_i \times \sum_{i=1}^{n} f(x_i)$$

$$= \frac{2}{n}\left[-\frac{4}{n^2}(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}) + \frac{2}{n}\left(\frac{n^2}{2} + \frac{n}{2}\right) + 2(n)\right]$$

Simplify: $-\frac{8}{3} + \frac{4}{n} + \frac{4}{3n} + 2 + \frac{2}{n} + 4$

Step 6:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i).\Delta x_i = -\frac{8}{3} + 2 + 4 = \mathbf{\frac{10}{3}}$$

If you can, test on your calculator.