α-MATHEMATICS

Alpha Mathematics
FINAL EXAM PAPER

22 October 2021 **Time: 3 hours**

Grade 12 **Total: 200 marks**

QUESTION 1 [20 MARKS]

- Answer this question **on the answer sheet** that is attached to the front, by marking A, B, C or D with an X (cross).
- Please **DO NOT** remove this page from the question paper.
- Each question counts 2 marks.

1.1 Given $f(x) = 2x^7 + 8x^6 - 17x^3 + x^2 + 6$. Which of the following numbers is, according to the **rational root theorem, NOT** a **POSSIBLE** zero of f:

(A) 1 **(B)** -2

(C) 3 **(D)** -4

1.2 How many terms will the expansion of $\left(2x - \frac{1}{x}\right)^{11}$ contain?

(A) 13 terms **(B)** 12 terms

(C) 11 terms **(D)** 10 terms

1.3 The third term of the expansion of the power series of $\sqrt[3]{1-2x}$ is:

(A) $\frac{-1}{9}x^2$ **(B)** $\frac{1}{9}x^2$

(C) $-\frac{4}{9}x^2$ **(D)** $\frac{4}{9}x^2$

1.4 Given $f(x) = x^4 - x^3 - 3x^2 + x + 2$. The gradient of the tangent to f at the point (2; 0) is:

(A) 9 **(B)** 8

(C) 1 **(D)** 0

1.5 Which statement is always TRUE:

(A) A stationary point of a graph is also a point of inflection.

(B) A point of inflection of a graph is also a stationary point.

(C) A tangent at any point to a graph which is concave down, will have a negative gradient.

(D) A tangent at any point to a graph which is decreasing, will have a negative gradient.

.1.6 Solve for x: $\frac{5}{|x-1|} < -1$

(A) $x < -4$ or $x > 6$ **(B)** $-4 < x < 6$

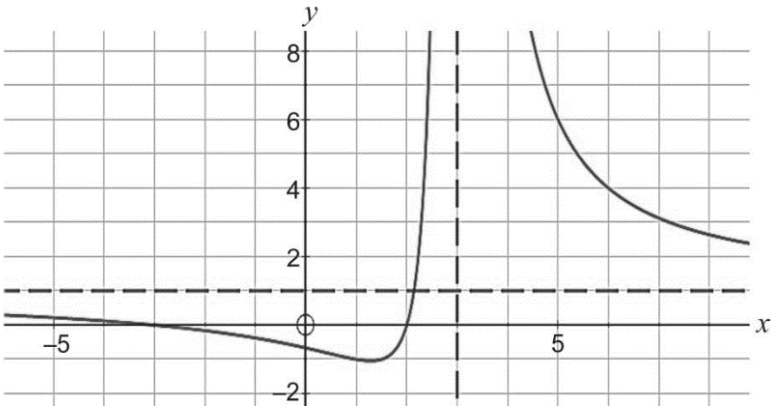
(C) No solution **(D)** $x \in \mathbb{R}$

1.7 If $f(x) = g(x) + 7$ for $x \in [0; 2]$, then $\int_0^2 (f(x) + g(x))\,dx =$

(A) $2\int_0^2 g(x)\,dx + 7$ **(B)** $2\int_0^2 g(x)\,dx + \frac{7}{2}$

(C) $2\int_0^2 g(x)\,dx + 14$ **(D)** $\int_0^2 g(x)\,dx + 14$

1.8 Look at the following graph. Which one of the equations can be the equation of this graph?



(A) $\frac{x^2-4}{x^2-9}$ **(B)** $\frac{x^2+x-6}{x-3}$

(C) $\frac{x-2}{x^2-x-6}$ **(D)** $\frac{x^2+x-6}{x^2-6x+9}$

1.9 Which of the following is equal to $\int_0^{\pi} \sin x\,dx$?

(A) $\int_0^{\pi} \cos x\,dx$ **(B)** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx$

(C) $\int_{-\pi}^{0} \sin x\,dx$ **(D)** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\,dx$

1.10 $F(x) = (f \circ g)(x)$, with $f(-2) = 8,\ f'(-2) = 4,\ f'(5) = 3,\ g(5) = -2,\ g'(5) = 6$. Determine the value of $F'(5)$.

(A) 24 **(B)** 8

(C) 12 **(D)** 20

QUESTION 2 [18 MARKS]

2.1 An air conditioner is switched on in a room. When the air conditioner is switched on, the equation $T(t) = 15 + 9e^{-0,3t}$ applies, where $T(t)$ is the temperature in $°C$ in the room t minutes after it was switched on.

(a) Determine the initial temperature in the room. (1)

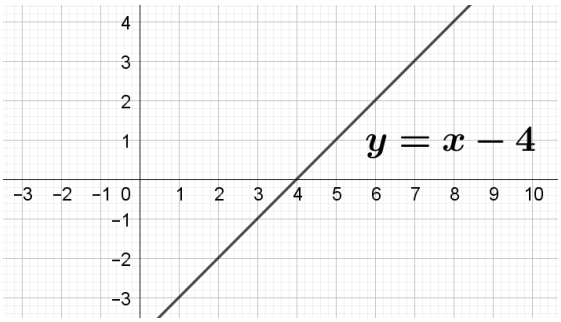
(b) What is the minimum temperature which the room can reach? (1)

(c) After how many minutes (rounded to the nearest minute) will the temperature in the room be 20 ℃ ? (4)

(d) Calculate the rate of change of the temperature 5 minutes after it was switched on. (3)

2.2 (a) The sketch below shows the graph of $y = x - 4$. Sketch the graph of $y = |x - 2| - 1$ on the sketch supplied. Clearly show all intercepts with the axes as well as the vertex. (4)

Draw your sketch on these axes:



(b) Show algebraically by using the definition that the equation $|x - 2| - 1 = x - 4$ will have no real roots. (Hence, solve for x.) (5)

QUESTION 3 [21 MARKS]

3.1 In the following calculation all angles must be given in terms of π and use surd form where applicable.

(a) Write $-\sqrt{3} + i$ in polar form. (2)

(b) Write $2 - 2\sqrt{3}i$ in polar form. (2)

(c) Use De Moivré's theorem and subsequently simplify the following:

$$\left(\frac{-\sqrt{3} + i}{2 - 2\sqrt{3}i}\right)^3$$

Give the final answer in rectangular form. (5)

3.2 The following system of equations is given:

$$2x + y + z = 0$$

$$x + ay - z = 5$$

$$4x + ay + 2z = 1$$

(a) Write the equations in matrix form: $Ax = B$ (2)

(b) (i) Calculate $|A|$, the determinant of A in terms of the given a. (2)

(ii) Use Cramer's rule and write down matrix A_y. (1)

(iii) Determine the value of a with Cramer's rule if it is also given that $|A_y| = 3$ and $y = 1$. (2)

3.3 Given $f(x) = \sin\left(x - \frac{\pi}{6}\right) + \frac{1}{2}$

(a) Determine the equation of $f^{-1}(x)$, the inverse function of f. (3)

(b) Give the range of f^{-1} . (2)

QUESTION 4 [20 MARKS]

4.1 $\boldsymbol{u} = (2; y; -1)$ and $\boldsymbol{v} = (0; -2; 5)$

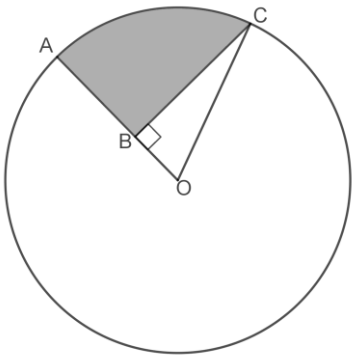
(a) Determine the value of y if a vector which is perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$ is equal to $18i - 10j - 4k$. (4)

(b) Accept that $y = 4$ for the following questions:

(i) Determine the area of the parallelogram which is formed by vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. (3)

(ii) Determine the angle between the two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. (6)

4.2 The sketch shows a circle with centre O and radius 10. A and C lie on the circumference of the circle. B lies on OA so that BC is perpendicular to AO. $\text{O}\hat{\text{C}}\text{B} = \frac{\pi}{6}$ radians.



(a) Give the size of $\text{A}\hat{\text{O}}\text{C}$ in terms of π. (1)

(b) Determine the area of the small sector AOC. (2)

(c) Hence, calculate the area of the shaded area ABC.
Leave the answer in surd form and in terms of π where applicable. (4)

QUESTION 5 [20 MARKS]

5.1 The term in the binomial expansion of $\left(2x + \frac{k}{x^2}\right)^6$, with k a constant, which is independent of x, is 2160. Determine the possible value(s) of k. (6)

5.2 Draw sketch graphs of functions with the following properties:

(a) $y = f(x)$ is continuous in the point $x = 1$, but not differentiable. (2)

(b) $y = g(x)$ has a removable discontinuity in the point $x = 1$ and $g(1)$ exists. (2)

5.3 Use mathematical induction and prove that

$$\sum_{r=1}^{n} r(r-1) = \frac{n(n^2-1)}{3}$$

(10)

QUESTION 6 [24 MARKS]

6.1 Use a Riemann sum and calculate the value of $\int_1^3 (2x^2)dx$. (11)

6.2 $(1 + 5\,\boldsymbol{i})$ is a zero of $f(x) = x^4 - 8x^3 + 47x^2 - 174x + 234$.

(a) Determine all possible real zeros of f. (7)

(b) The point $x = 3$ is a stationary point of the polynomial $f(x)$.
Use the second derivative and determine if it is a point of inflection.
If not, determine which type of stationary point it is. (6)

QUESTION 7 [17 MARKS]

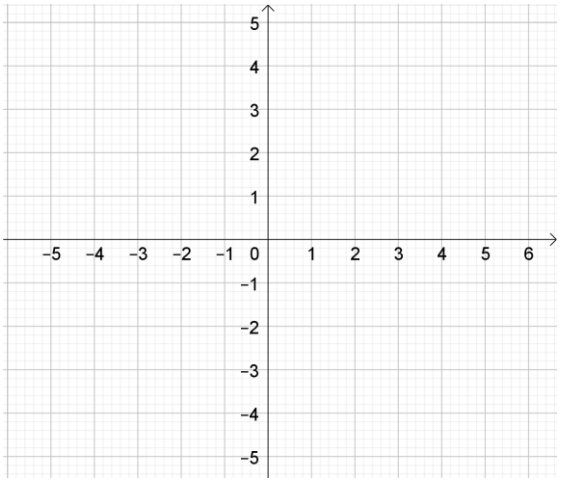
Given: $f(x) = \frac{x^2+2x+2}{x+1} = x + 1 + \frac{1}{x+1}$

7.1 Give the intercept of f with the y-axis. (1)

7.2 Give the equations of the asymptotes of f and say which type of asymptote they are.

7.3 Calculate the co-ordinates of the turning points of f. (6)

7.4 Make a sketch graph of f on the axes below. Clearly show the asymptotes and the turning points. (6)



QUESTION 8 [21 MARKS]

8.1 Differentiate the following functions:

(a) If $f(x) = 5^x \times \log x$, determine $f'(x)$ (3)

(b) $D_x\left[\tan\left(e^{\frac{x}{2}+1}\right)\right]$ (4)

8.2 Use implicit differentiation and determine the value of the gradient of the tangent to $x^2.\ln(y) + 2x - 5y = 1$ in the point (3; 1). (6)

8.3 The graph of $y = -\frac{1}{2}x$ and $y = \ln(x) - 4$ intersect at A.

(a) Show algebraically that this point lies in the interval $[4; 5]$. (3)

(b) Hence, use Newton's method and calculate the x-value of this point, correct tot 4 decimal digits. Clearly show how you use Newton's method. (5)

QUESTION 9 [20 MARKS]

9.1 Determine $\int (\cot^2(3x) - \sin(2x))dx$ (5)

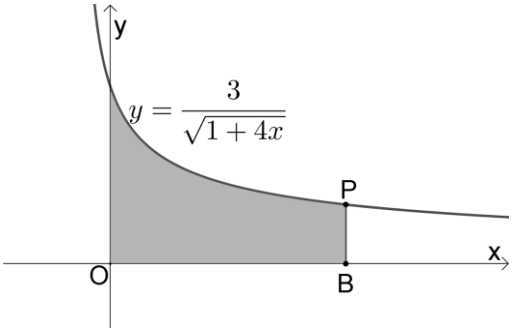
9.2 Use integration by parts and determine $\int_0^{ln(a)} xe^x dx$, with a a constant. Simplify the answer. (5)

9.3 Determine $\int (\frac{cx^2}{1+x^3})dx$ with c a constant. (3)

9.4 Determine $\int \frac{-4x^2+2x-102}{(x^2+25)(x-1)}dx$ by using partial fractions. (7)

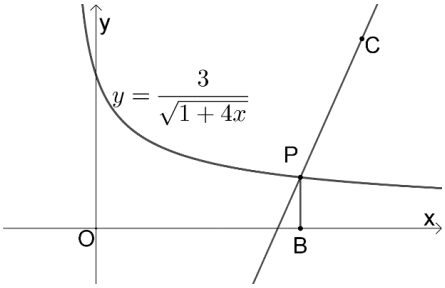
QUESTION 10 [19 MARKS]

The sketch shows the graph of $y = \frac{3}{\sqrt{1+4x}}$, for $x \geq 0$ with the point P on the graph and B on the x-axis. PB is perpendicular to the x-axis.



10.1 The area of the shaded region is equal to 3.
Determine the x-value of the point P. (7)

10.2 PC is the normal to the graph at P (thus PC is perpendicular to the tangent at P).



Calculate the y-intercept of PC. Accept B(2; 0). (6)

10.3 The graph $y = f(x)$ rotates around x-axis between $x = 1$ and $x = 2$.
The volume of the rotating body is equal to $\boldsymbol{\theta \ln \beta}$.
Determine the values van θ and β. (6)

