α-MATHEMATICS

Alpha Mathematics
PRELIM EXAM PAPER

September 2020 **Time: 3 hours**

Grade 12 **Total: 200 marks**

INSTRUCTIONS AND INFORMATION

Carefully read through the following instructions before answering the paper:

1. Answer ALL 10 questions on this examination paper.
2. Write your name and ID number on the front page of the answer sheet.
3. Non-programmable calculators may be used, unless otherwise indicated in a specific question.
4. Unless indicated otherwise, all answers, where necessary, must be given correct to two decimal digits.
5. The diagrams on the question paper are not necessarily drawn to scale.
6. All angles are given in radians. Answers must be given in radians where applicable.
7. This examination paper consists of an examination paper of 8 pages, a formula sheet of 3 pages and an answer sheet of 2 pages.
8. Question 1 consists of 10 multiple choice questions. Answer it on the answer sheet.
9. For all other questions, all necessary calculations must be shown clearly.
   The correct answer on its own will not necessarily lead to full marks.
10. Write neatly and legibly.

QUESTION 1 [20 MARKS]

- Answer this question **on the answer sheet,** which is attached to the front, by making a X (cross) on A, B, C or D.
- Each question counts 2 marks.

1.1 If $f(x) = \begin{cases} 2x & if \quad x \leq 1 \\ \log x & if \quad x > 1 \end{cases}$, then

(A) f is continuous in the point $x = 1$

(B) f is differentiable in the point $x = 1$

(C) there exists a removable discontinuity at $x = 1$

(D) there exists a jump discontinuity at $x = 1$

1.2 Given $f(x) = cotx$. If $f'(x) = -2$, then $x =$

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{4}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{\pi}{2}$

1.3 The graph of $y = |4 - 2x| - 2$ has a salient point at

(A) $(-4; -2)$ **(B)** $(4; -2)$

(C) $(2; -2)$ **(D)** $(-2; -2)$

1.4 Which of the following expressions' power range will converge if $|x| < 2$?

(A) $\frac{1}{2+x}$ **(B)** $\frac{1}{1+2x}$

(C) $\frac{1}{1-2x}$ **(D)** $\frac{1}{2x-2}$

1.5 If $\cos x = e^y$, then $\frac{dy}{dx} =$

(A) $\frac{1}{\tan x}$ **(B)** $-\frac{1}{\tan x}$

(C) $\tan x$ **(D)** $-\tan x$

1.6 The equation of the tangent at $y = arc\tan\left(\frac{x}{2}\right)$ at the origin is

(A) $y = \frac{x}{2}$ **(B)** $y = x$

(C) $x = 0$ **(D)** $y = 0$

1.7 The maximum value of $\frac{lnx}{x}$ is at $x =$

(A) 1 **(B)** $\frac{3}{2}$ **(C)** $e^{\left(\frac{1}{2}\right)}$ **(D)** e

1.8 $\int \cos(2x + 3)\, dx$ =

(A) $-\frac{1}{2}\sin(2x + 3) + c$ **(B)** $\frac{1}{2}\sin(2x + 3) + c$

(C) $-\frac{3}{2}\sin(2x + 3) + c$ **(D)** $\frac{3}{2}\sin(2x + 3) + c$

1.9 If $\frac{dy}{dx} = 4x$ and $y = 4$ when $x = 0$, then $y =$

(A) $4x + 4$ **(B)** $4 + x^2$

(C) $2x^2 + 4$ **(D)** $4 + 4x^2$

1.10 $(f \circ g)(x) = x^2 + 2x$. If $g(2) = 4$ and $g'(2) = 3$, determine $f'(4)$

(A) 1 **(B)** 2 **(C)** 5 **(D)** 6

Answer the following questions **on the answer sheets.**

QUESTION 2 [23 MARKS]

2.1 The population of sardines can be determined with the equation

$$P(t) = 20 - (2 \times 4^{0,1t})$$

where P(t) is the population in millions after t weeks.

(a) Determine the initial population, when $t = 0$. (2)

(b) Calculate after how many weeks there should be 10 million sardines. (4)

(c) Calculate the rate of change of sardines during the 10’th week and state whether the population increases or decreases. (5)

2.2 Solve for x: $|x - 5| = 2x - 1$. (6)

2.3 Determine the sixth term in the binomial expansion of $\left(-\frac{x}{3} + \frac{2}{x^2}\right)^9$. (6)

QUESTION 3 [20 MARKS]

3.1 The equation $x^4 - 2x^3 - 6x^2 + 16x - 16 = 0$ has a root $x = 1 + i$. Solve the equation fully in $\mathbb{C}$. (8)

3.2 (a) Given $(-1 + i)^6$. Use de Moivre’s theorem, with angles in terms of π and show that the expression is always non real. (6)

(b) Simplify:

$$\frac{(-1 + i)^6}{\sqrt{3} - i}$$

Work in polar- or exponential form. Use surds and π if necessary. Give the answer in rectangular form. (6)

QUESTION 4 [15 MARKS]

4.1 Given $\boldsymbol{u} = 2i + aj - 2k$ and $\boldsymbol{v} = 4i - 2j + k$.

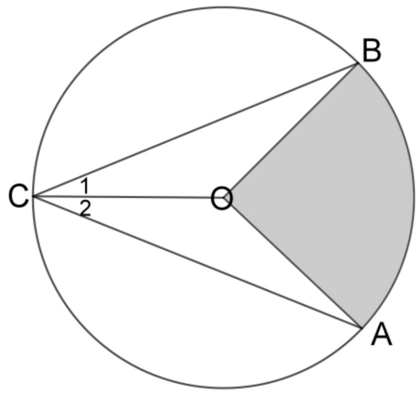
(a) Determine the value(s) of a if the magnitude of $\boldsymbol{u}$ equals 3. (4)

(b) If $a = -1$, determine the magnitude of the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. (5)

4.2 Given $\boldsymbol{a} = (-2; -1; 4)$ and $\boldsymbol{b} = (0; -5; 2)$. Determine a vector that is perpendicular to these two vectors. (6)

QUESTION 5 [19 MARKS]

5.1 The sketch shows circle CAB with centre O and radius 5 units. Radï OB, OA and OC is drawn as well as cords AC and BC. $\hat{C}_1 = \hat{C}_2 = \frac{\pi}{6}$. Sector OAB is shaded.



(a) Determine the circumference of sector OAB. (3)

(b) Determine the area of sector OAB. (3)

5.2 **Use the diagram on the answer sheet** and make a sketch graph of $y = -arcsin(x - \frac{1}{2})$. Clearly show the x- and y-intercept. (5)

5.3 Given the set of equations:

$$\begin{aligned} 2x \quad +y \quad +z \quad &= 0 \\ x \quad -2y \quad -z \quad &= \boldsymbol{p} \\ 4x \quad +3y \quad +2z \quad &= 1 \end{aligned}$$

The value of x = 2. Use Cramer's method and determine the value of $\boldsymbol{p}$**.** Accept that

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 4 & 3 & 2 \end{vmatrix} = 3$$

Clearly show the determinants that you use. (8)

QUESTION 6 [23 MARKS]

Given the graph with equation $f(x) = \frac{x^2-2x-3}{x-1}$.

6.1 Calculate the x- and y-intercepts (4)

6.2 Determine the equations of all asymptotes of f. (4)

6.3 What is the type of discontinuity at the vertical asymptote called?
Give a reason. (2)

6.4 Show that f has no stationary points. (7)

6.5 **Use the diagram sheet on the answer sheet** and make a sketch graph of f.
Clearly show the intercepts with the axes and the asymptotes on your sketch. (6)

QUESTION 7 [19 MARKS]

7.1 Use mathematical induction and prove that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(9)

7.2 Use a Riemann sum and calculate the value of $\int_1^3 3x^2\, dx$. (10)

QUESTION 8 [22 MARKS]

8.1 Given the function: $f(x) = \begin{cases} \arctan x + 1 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x = 0 \\ x + 1 & \text{if} \quad x > 0 \end{cases}$

(a) Show that $f(x)$ is continuous for $x = 0$. (4)

(b) Determine if $f(x)$ is differentiable at $x = 0$. (4)

8.2 Differentiate the following as asked:

(a) Determine $f'(x)$ if $f(x) = lnx \times \log x$. (3)

(b) $D_x[\sec\left(\frac{2}{x}\right)]$ (3)

(c) If $y = xe^x + ye^y - e$, use implicit differentiation and determine $\frac{dy}{dx}$. (8)

QUESTION 9 [18 MARKS]

Determine the following integrals

9.1 $\int (2x^2 + 2^x + e^{-x})dx$ (4)

9.2 $\int \frac{x^2-5x+6}{(x^2+1)(x-1)}\, dx$ by using partial fractions. (9)

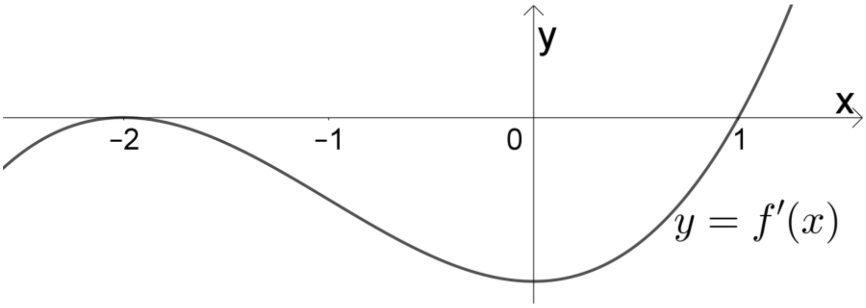
9.3 $\int \sin^5(5x) . \cos(5x)\, dx$ You may use substitution. (5)

QUESTION 10 [21 MARKS]

10.1 The function $f(x) = \sqrt{2x} - \sin x - 2$ has a zero in the interval $[2; 3]$.
Use Newton's method and determine the value of this zero
correct to 4 decimal digits. Show clearly how you use Newton's method. (4)

  Please turn over

10.2 The sketch beneath shows the graph of $\boldsymbol{y = f'(x)}$**,** the derivative of $y = f(x)$. This graph has a local maximum turning point at $(-2; 0)$ and it intercepts the x-axis at $x = 1$. It has a local minimum turning point where $x = 0$.



Motivate your answer each time by referring to f' and/ or f''.

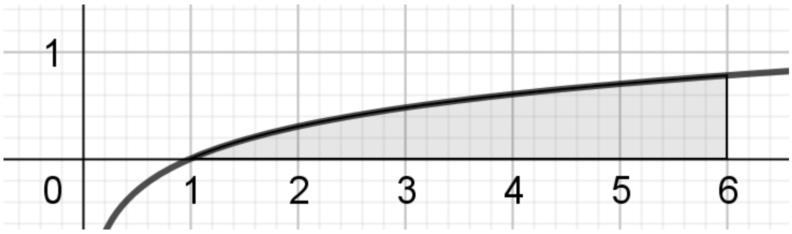
At which value(s) of x will $\boldsymbol{y = f(x)}$

(a) have a turning point? State whether it is a maximum of minimum. (3)

(b) has a point of inflection that is also a stationary point? (4)

(c) be concave down? (3)

10.3 The sketch shows the graph of $f(x) = lnx$, with the area between the x-axis, the graph and the line $x = 6$ shaded. Use **factor integration** and determine the area of this region. Show all your integration steps, otherwise marks will not be given. Give the answer in the form $alnb + c$.



(7)