α-MATHEMATICS

Alpha Mathematics
FINAL EXAM PAPER

21 October 2019 **Time: 3 hours**

Grade 12 **Total: 200 marks**

INSTRUCTIONS AND INFORMATION

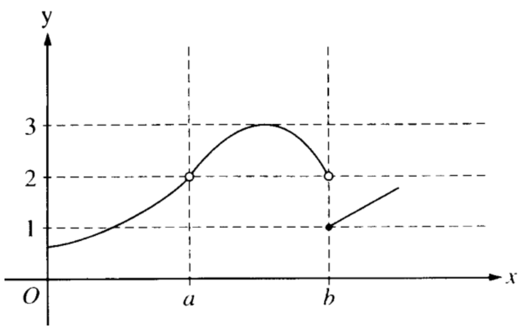
Carefully read through the following instructions before answering the question paper:

1. Answer all 10 questions on this exam paper.
2. Write your name and ID number on the front page of the question paper.
3. Non-programmable calculators may be used, unless otherwise indicated at a specific question.
4. Unless indicated otherwise, all answers, where applicable, must be given correct to two decimal places
5. The diagrams in the question paper are not necessarily drawn to scale.
6. All angles are given in radians. Answers must be given in radians where applicable.
7. This question paper consists of a front page, 26 pages and a formula sheet of 3 pages.
8. Question 1 consists of 10 multiple choice questions. Answer it on the answer sheet. This answer sheet is on the front of the paper.
   Do not remove the answer sheet from the question paper.
9. For all other questions, all necessary calculations must be shown clearly. The correct answer on its own will not necessarily lead to full marks.
10. Additional writing space is provided at the end of this question paper. Clearly indicate if you make use of this to complete a question.
11. Write neatly and legibly.

QUESTION 1 [20 MARKS]

- Answer this question **on the answer sheet,** that is attached to the front, by marking a X (cross) on A, B, C or D.
- Please **DO NOT** remove this page from the question paper.
- Each question counts 2 marks.

1.1 The sketch shows a graph of the function f:



Which of the following statements are true for f?

(A) f is continuous at $x = a$ **(B)** $\lim_{x\to a} f(x) = \lim_{x\to b} f(x)$

(C) $\lim_{x\to a} f(x) = 2$ **(D)** $\lim_{x\to b} f(x) = 1$

1.2 The temperature in a room is given by an equation $H(t)$ where H is the temperature in degrees Celsius, t minutes after the air conditioner is switched on. Which of the following is the best interpretation for $\boldsymbol{H'(5) = 2}$?

(A) The temperature in the room is 2°C, 5 minutes after switching on.

(B) The temperature in the room increases with 2°C during the first 5 minutes.

(C) The temperature in the room increases at a constant tempo of $\frac{2}{5}$ °C per minute.

(D) The temperature in the room increases with a tempo of 2°C per minute, 5 minutes after switching on.

1.3 Which of the following distractors will be the value of one of the discriminants which are used to solve the following system of equations by using Cramer's rule?

$$\begin{aligned} x + 2y &= -4 \\ 3x - 2y &= 8 \end{aligned}$$

(A) 4 **(B)** -2 **(C)** 20 **(D)** 8

1.4 If the vectors $2i + 3j - k$ and $i + ak$ are perpendicular, then the value of $a =$

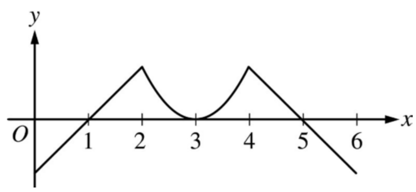
(A) 2 **(B)** 5

(C) -1 **(D)** None of these

1.5 $\int_1^e \left(\frac{x^2-1}{x}\right) dx =$

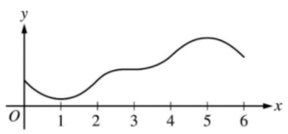
(A) $\frac{e^2}{2} + \frac{1}{2}$ **(B)** $\frac{e^2}{2} - \frac{3}{2}$ **(C)** $\frac{e^2}{2} - 2$ **(D)** $\frac{e^2}{2} - e$

1.6 The graph of $y = f'(x)$, the derivative of $y = f(x)$, is shown.

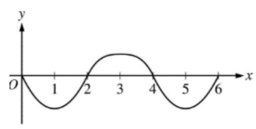


Which of the following can be a sketch of the function $y = f(x)$?

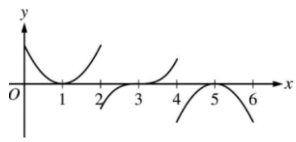
(A)



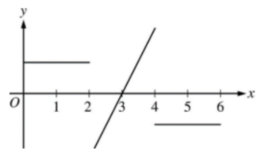
(B)



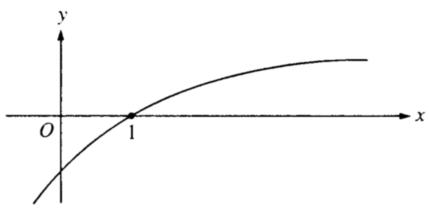
(C)



(D)



1.7 The sketch shows the graph of $y = f(x)$ for which f' and f'' exist.



Which statement is true?

(A) $f(1) < f'(1) < f''(1)$ **(B)** $f(1) < f''(1) < f'(1)$

(C) $f'(1) < f(1) < f''(1)$ **(D)** $f''(1) < f(1) < f'(1)$

1.8 If $h(x) = f(g(x))$, then $h''(x) =$

(A)	$f''(g(x))g'(x) + f'(g(x))g''(x)$	**(B)**	$f''(g(x))[g'(x)]^2$
(C)	$f''(g(x))[g'(x)]^2 + f'(g(x))g''(x)$	**(D)**	$f''(g(x))g''(x)$

1.9 The equation of the tangent to the graph $y = \cos(2x)$ at the point $x = \frac{\pi}{4}$ is:

(A)	$y - 1 = -(x - \frac{\pi}{4})$	**(B)**	$y = -2(x - \frac{\pi}{4})$
(C)	$y - 1 = -2(x - \frac{\pi}{4})$	**(D)**	$y = -(x - \frac{\pi}{4})$

1.10 If f is a continuous function and $F'(x) = f(x)$ for all real values of x, then $\int_2^3 f(2x)\,dx =$

(A)	$2\text{F}(3) - 2\text{F}(2)$	**(B)**	$\frac{1}{2}F(3) - \frac{1}{2}F(2)$
(C)	$2\text{F}(6) - 2\text{F}(4)$	**(D)**	$\frac{1}{2}F(6) - \frac{1}{2}F(4)$

Answer the following questions **on the exam paper** on the lines provided at the end of each question. Clearly indicate if you use the additional writing space at the end of the paper to complete a question.

QUESTION 2 [19 MARKS]

2.1 The proportion of radioactive carbon-14 isotopes (C-14) after t years to the initial amount of C-14 in a fossil is given by: $\frac{A(t)}{A_0} = e^{\left(\frac{-1.21t}{10\,000}\right)}$.

(a) Calculate the proportion $(\frac{A(t)}{A_0})$ radioactive C-14 in a fossil after 1 000 years. (2)

(b) If the proportion $(\frac{A(t)}{A_0})$ in a fossil is 0,75 after t years, determine t. (3)

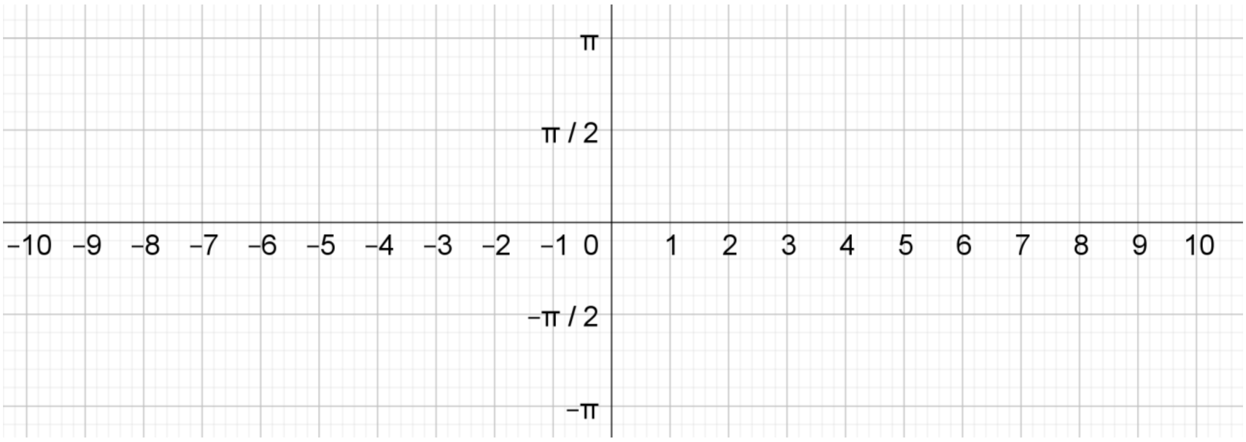
(c) Determine the half-life of carbon-14, meaning the time it takes for the radio activity of carbon-14 to halve.
Round off the answer to the nearest 10. (3)

2.2 Solve for x: $|2x + 6| = -2x + 3$ (6)

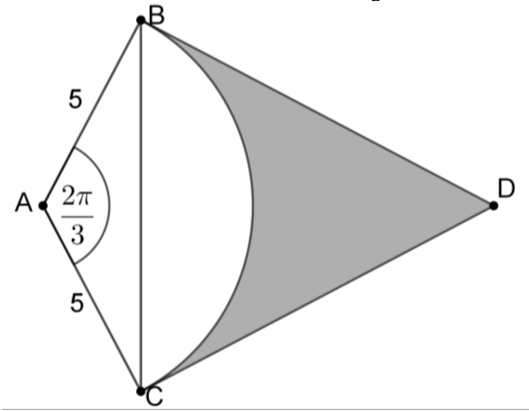
2.3 Use the binomial theorem and determine the term in the expansion of $\left(\frac{x^3}{2} - \frac{2}{x}\right)^8$ that is independent of x. (5)

QUESTION 3 [21 MARKS]

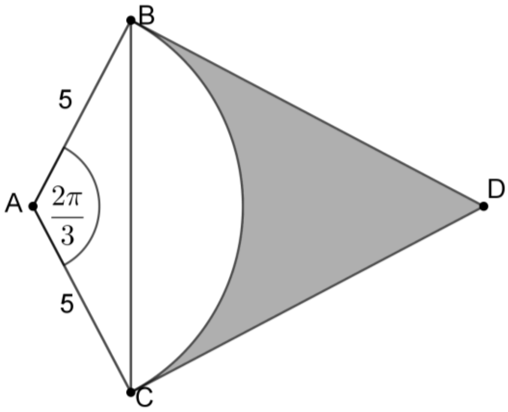
3.1 Draw a neat sketch graph of $f(x) = \arctan(2x) - \frac{\pi}{4}$.
Clearly show the intersections with the axes and the asymptotes on your graph. (5)



3.2 In the diagram BD and CD are tangents to a circle with centre A.
AB and AC are radii with a length of 5. $B\hat{O}C = \frac{2\pi}{3}$ radians.



(a) Show that BD = CD = $5\sqrt{3}$. (3)



(b) Calculate the area of the shaded section, with BD = CD = $5\sqrt{3}$. (4)

3.3 (a) Write $-\sqrt{3} + i$ in polar form. The angle must be given in terms of π. (4)

(b) Hence calculate the value of $\frac{(-\sqrt{3}+i)^{12}}{32cis\left(\frac{\pi}{3}\right)}$.
Use de Moivre's theorem and give your answer still in polar form. (3)

(c) Convert the answer to rectangular form. (2)

QUESTION 4 [18 MARKS]

4.1 (a) Give a condition under which a n^th^-degree polynomial in $\mathbb{Q}[x]$ will have at least one real root. Motivate your answer. (2)

(b) Given $P(x) = x^4 + 2x^3 + 18x^2 + 2x + 17$ with a root $x = -1 + 4i$.
Determine the real roots of P, if any. (6)

4.2 Use mathematical induction as well as the formulae for compound angles and prove de Moivre's theorem if $r = 1$.
Thus prove that $(\boldsymbol{\cos\theta} + \boldsymbol{i\sin\theta})^{\boldsymbol{n}} = \boldsymbol{\cos}(\boldsymbol{n\theta}) + \boldsymbol{i\sin}(\boldsymbol{n\theta})$**.** (10)

QUESTION 5 [21 MARKS]

5.1 Given vectors $\boldsymbol{u} = (1; a; 4)$ and $\boldsymbol{v} = (4; -1; -1)$. The distance between the end points of the vectors is $5\sqrt{2}$. Calculate the value of a if $a < 0$. (4)

5.2 Given the vectors $\boldsymbol{a} = (1; 3; 5)$, $\boldsymbol{b} = (2; -1; 0)$ and $\boldsymbol{c} = (-3; 0; 1)$.
(a) Determine the angle between $\boldsymbol{a}$ and the $\boldsymbol{z}$-axis. (3)

(b) Determine the area of the parallelogram formed by $\boldsymbol{b}$ and $\boldsymbol{c}$. (5)

(c) Determine the angle between $\boldsymbol{a} = (1; 3; 5)$ and $\boldsymbol{b} = (2; -1; 0)$. (4)

5.3 The eigen values $\boldsymbol{k}$ of a matrix A can be determined with the determinant equation:
$|\boldsymbol{A} - \boldsymbol{kI}| = \boldsymbol{0}$ where ***I*** is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, for a 2x2 matrix.

Determine the eigen value(s) $\boldsymbol{k}$ of the matrix $\begin{pmatrix} -5 & 2 \\ -9 & 6 \end{pmatrix}$.

(5)

QUESTION 6 [19 MARKS]

6.1 Solve for a and b if f is differentiable for all real values of x, with (6)

$$f(x) = \begin{cases} ax + 2b - 4 & as \quad x \le 2 \\ bx^2 - ax + a & as \quad x > 2 \end{cases}$$

6.2 Differentiate the following functions as asked. The answers need not be simplified.

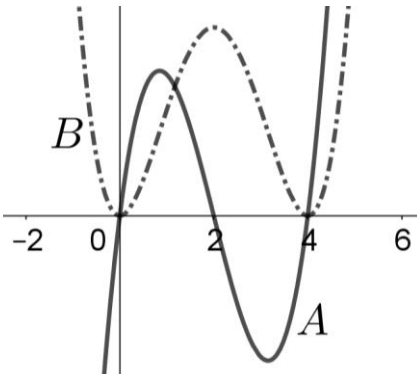
(a) Determine $f'(x)$ if $f(x) = \log(x^2) \times \arctan(2x)$. (5)

(b) $D_x\left[\cos\sqrt{x}\right]$. (3)

(c) Use implicit differentiation and determine $\frac{dy}{dx}$ if $e^y = 2^{xy}$. (5)

QUESTION 7 [21 MARKS]

7.1 The following graph shows the sketch of $y = f'(x)$ and $y = f''(x)$, the first and second derivative of f.

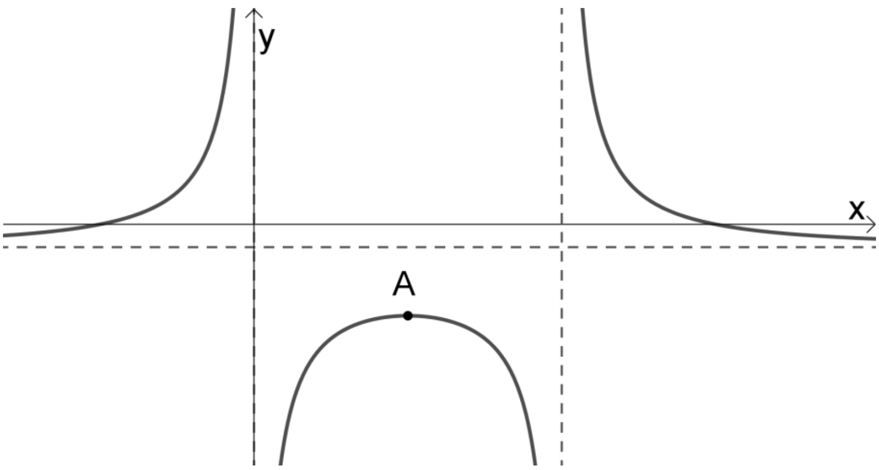


(a) Which of the graphs, A and B, is the graph of $f'(x)$ and $f''(x)$? (2)

(b) Give the x-value(s) of the point(s) of inflection of f.
Motivate your answer. (3)

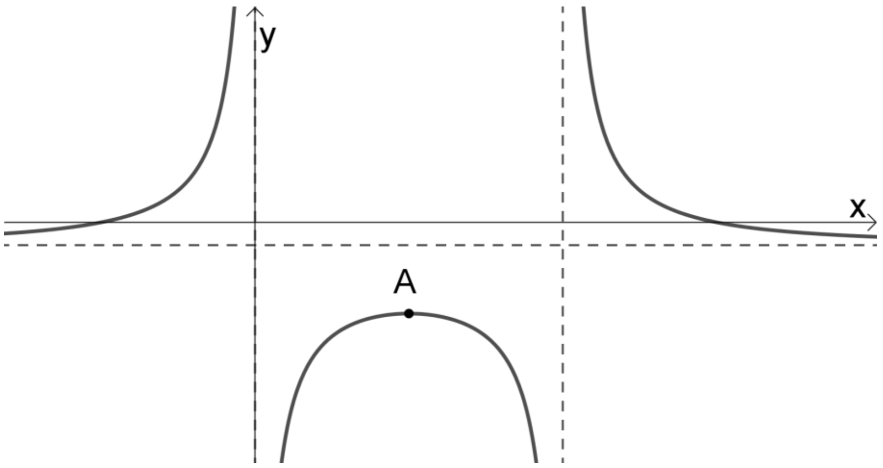
(c) Give the x-value(s) of the point(s) of inflection that is/are also stationary points of f. Motivate your answer. (2)

7.2 The sketch shows the graph of $f(x) = \frac{x^2-2x-3}{2x-x^2}$.

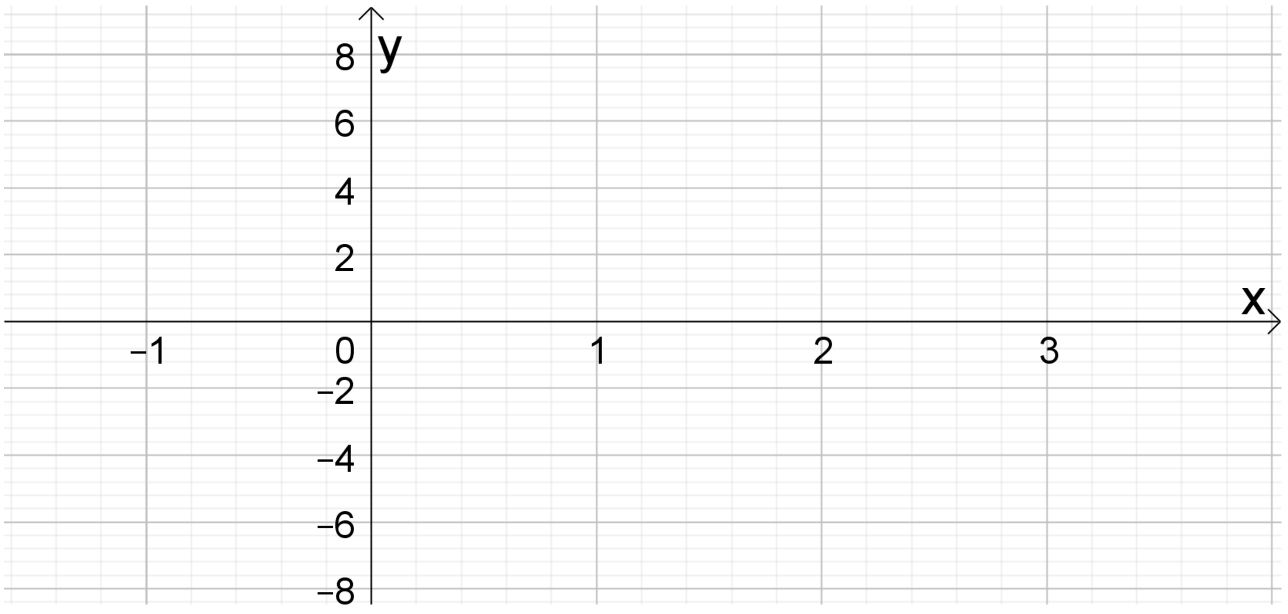


(a) Determine the equations of all asymptotes of f. (3)

(b) The graph has one maximum turning point at A.
Determine the coordinates of A. (6)



(c) Draw a neat sketch graph of $g(x) = \left|\frac{x^2-2x-3}{2x-x^2}\right|$.
Clearly show all intercepts with the axes, asymptotes and the coordinates of the turning point. (5)



QUESTION 8 [20 MARKS]

8.1 Determine the following integrals:

(a) $\int (\cot^2 2x + \cos^2 x)dx$ (5)

(b) $\int \frac{dx}{\sqrt{-4x^2-4x}}$ (5)

8.2 (a) Decompose $\frac{1}{x^2+3x+2}$ into partial fractions. (5)

(b) **Hence** determine the value of A if $\int_1^2 \frac{1}{x^2+3x+2} dx = lnA$. (5)

QUESTION 9 [22 MARKS]

9.1 (a) Use a Riemann sum and determine $\int_0^2 x(x-2)dx$. (9)

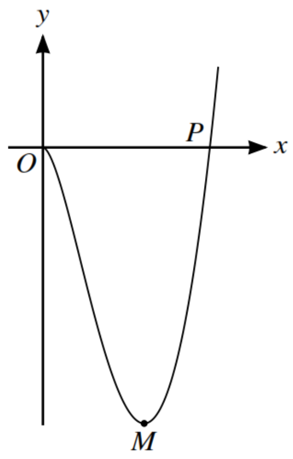
(b) Now determine the area between $y = x(x-2)$ and the x-axis. (1)

9.2 The graph of $y = e^x$ rotates around the x-axis. The volume of the body of revolution that is thus formed between the y-axis and the line $x = ln\ (p)$ with $p > 0$, is equal to $\frac{15\pi}{2}$. Determine the value of p. (6)

9.3 The graphs of $y = x - 2$ and $y = \cot\left(-\frac{x}{2}\right)$ intersect each other between $x = 5$ and $x = 6$. Use Newton's method and determine this intercept correct to four decimal digits. Clearly show how you use Newton's method. (6)

QUESTION 10 [19 MARKS]

10.1 The sketch shows the graph of $y = 3x^2 \ln\frac{x}{6}$.



Show that the gradient of the graph at the point P, the x-intercept of the graph, is equal to three times the value of this x-intercept. (7)

10.2 (a) Show that $\frac{d}{dx}(x - \arctan x) = \frac{x^2}{1+x^2}$. (3)

(b) **Hence** calculate $\int_0^{\sqrt{3}} x.\arctan x\, dx$. Give the answer in terms of π and use root form f necessary. Use factor integration. (9)

